ESAT Guide

Mathematics 1

June 2026

Contents

INTRODUCTION

This is worth reading before you use this guide.

We have three aims in writing this guide:

First, we want to set out what we expect you to know for the ESAT. We do this by basing each part of the guide on the relevant part of the specification.

Second, we want to encourage you to think deeply and carefully about science and mathematics and to develop a good understanding of the topics in the specification. To help with this, we have added a lot of discussion and examples as well as some exercises throughout the guide.

Third, we want to make sure that all candidates have access to a free resource to help them prepare for the ESAT.

How to use this guide

You do not need to work through all this guide as you will find that you know many of the topics in the specification very well already. Use this guide as a resource to help you clarify and review topics that you are less familiar with. We have broken down our discussion to fit exactly with the specification to make things as simple to navigate as possible.

What this guide is not

This guide is not a comprehensive textbook: we do not cover every topic to the same level of detail, and we do not develop every topic from scratch. It is also not a substitute for sustained hard work and preparation. It is a resource to help you and to guide you in the right direction.

Should I take an ESAT course?

We do not recommend that you take a course, and we do NOT endorse any courses. No one from the ESAT development team teaches on any courses. All the resources you need to prepare are available from the UAT-UK website and are entirely free.

A final note

We have used boxes throughout the guide to help you navigate.

The relevant part of the specification is found in these sorts of boxes:

Specification

examples in these sorts of boxes:

Examples

and exercises [answers are at the end of each section] in these sorts of boxes:

Exercises

We hope to be able to update and, if necessary, correct the guide now and again. Look at the date on the front page to see when the guide was last edited.

M1. Units

M1.1

Use standard units of mass, length, time, money and other measures.

Use compound units such as speed, rates of pay, unit pricing, density and pressure, including using decimal quantities where appropriate.

Standard units

The commonly used standard units are:

Mass

- Milligrams (mg)
- Grams (g)
- Kilograms (kg)
- Tonnes (t)

Force

- Newtons (N)

Length

- Millimetres (mm)
- Centimetres (cm)
- Metres (m)
- Kilometres (km)

Area

- Square millimetres (mm^2)
- Square centimetres (cm^2)
- Square metres (m^2)
- Square kilometres (km^2)

Capacity

- Millilitres (ml)
- Litres (l)

Volume

- Cubic millimetres (mm^3)
- Cubic centimetres (cm^3)
- Cubic metres (m^3)
- Millilitres (ml)
- Litres (l)

Time

- Seconds
- Minutes
- Hours
- Days
- Weeks
- Months
- Years

Capacity and volume

1 ml = 1 cm^3

1 l = 1 dm^3 = 1000 cm^3

1000 l = 1 m^3

Small quantities of liquid (e.g. drinks, medicines) are measured in ml and l. Large quantities (e.g. swimming pools, reservoirs) are measured in m^3.

Time

A year is 12 months – 365 days in a normal year but 366 in 'leap years'. Leap years occur nearly every 4 years.

A century is 100 years. A millennium is 1000 years.

Exercise A Standard units

Which units are commonly used to measure:

1 The volume of water in a swimming pool?

2 The area of a kitchen floor?

3 The volume of liquid in a can of cola?

Answers to exercises are at the end of this section

Compound units

Compound units are formed when two quantitative forms of measurement need to be combined (such as *metres* and *seconds* into *metres per second*).

For example, to find the average speed of a car, distance travelled in km is divided by the time taken in hours. The km are divided by hours so the units are km/h and read as kilometres per hour.

An alternative way of writing this type of compound unit is km h^{-1}

Exercise B Compound units

Find the missing units:

1. The density of a piece of rock of mass 600 g and volume $20cm^3$ is 30_____
2. The average speed of a ball which travels 10 m in 2 seconds is 5_____
3. The average rate of pay for a worker who is paid £300 for 15 hours of work is 20_____
4. The average speed of a car which travels 200 km in 4 hours is 50_____
5. The pressure exerted by a force of 6 N on a board of area 2 m^2 is 3_____

Answers to exercises are at the end of this section

Unit cost of an item

If x items cost £y in total, then the unit cost is the cost of 1 item which is found by dividing the total cost by the number of items. This is £ $\frac{y}{x}$ per item.

Example: Unit cost of an item

50 boxes of sweets cost £215. What is the unit cost of a box of sweets?

Answer:

The unit cost is the cost of 1 box.

1 box costs £$\frac{215}{50}$ = £4.30 per box.

M1.2

Change freely between related standard units (e.g. time, length, area, volume/capacity, mass) and compound units (e.g. speed, rates of pay, prices, density, pressure) in numerical and algebraic contexts.

Changing between standard units

The table shows the conversion rates for common measures.

Length	1 km = 1000 m 1 m = 100 cm = 1000 mm 1 cm = 10 mm	
Area	$1\text{ km}^2 = 1000^2\text{ m}^2 = 1\,000\,000\text{ m}^2$ $1\text{ m}^2 = 100^2\text{ cm}^2 = 10\,000\text{ cm}^2$ $1\text{ m}^2 = 1000^2\text{ mm}^2 = 1\,000\,000\text{ mm}^2$ $1\text{ cm}^2 = 10^2\text{ mm}^2 = 100\text{ mm}^2$	
Volume / capacity	1 l = 1000 ml $1\text{ ml} = 1\text{ cm}^3$ $1\text{ l} = 1000\text{ ml} = 1000\text{ cm}^3$ $1000\text{ l} = 1\text{ m}^3$	$1\text{ km}^3 = 1000^3\text{ m}^3 = 1\,000\,000\,000\text{ m}^3$ $1\text{ m}^3 = 100^3\text{ cm}^3 = 1\,000\,000\text{cm}^3$ $1\text{ m}^3 = 1000^3\text{ mm}^3 = 1\,000\,000\,000\text{ mm}^3$ $1\text{ cm}^3 = 10^3\text{ mm}^3 = 1000\text{ mm}^3$
Mass	1 kg = 1000 g 1 g = 1000 mg	
Time	60 seconds = 1 minute 60 minutes = 1 hour 24 hours = 1 day 7 days = 1 week	12 months = 1 year 100 years = 1 century 1000 years = 1 millennium

Exercise C Changing between standard units

How many cm^2 are in 2.65 m^2?

Answers to exercises are at the end of this section

Changing between compound units

To change between compound units, change each of the units involved separately.

Example: Changing between compound units

The density of a piece of metal is 20 g cm^{-3}. What is this as a density in kg m^{-3}?

We can do this as follows:

First think of 20 g cm^{-3} as 20g per 1 cm^3 so we can write this as follows:

$$20 \text{ g cm}^{-3} = \frac{20 \text{ g}}{1 \text{ cm}^3}$$

We then convert g to kg and cm^3 to m^3 as follows:

$$\frac{20 \text{ g}}{1 \text{ cm}^3} = \frac{\frac{20}{1000}\text{kg}}{\frac{1}{1000000}\text{m}^3} = \frac{20}{1000} \div \frac{1}{1000000} = \frac{20}{1000} \times \frac{1000000}{1} = 20\,000 \text{ kg m}^{-3}$$

Here is another way to approach this example:

First, convert the g to kg. The value of the density in kg/cm^2 will be smaller than the value of the density in g/cm^2 so divide by the conversion factor:

$$20 \text{ g/cm}^3 = \frac{20}{1000} \text{ kg/cm}^3 = 0.02 \text{ kg/cm}^3$$ (to convert from g to kg divide by 1000)

Next, convert the cm^3 to m^3. The value of the density in kg/cm^3 will be greater than the value of the

density in kg/cm^3 so multiply by the conversion factor: 0.02 kg/cm^3 = 20 000 kg/m^3

(to convert from m^3 to cm^3 multiply by 1 000 000)

Example: Units and Problem solving

A car travels 35 000 m in 30 minutes. What is its average speed in km/h?

We want the speed in km h^{-1}, so start by converting the distance in metres to a distance in kilometres.

35 000 m = $\frac{35000}{1000}$ km = 35 km

Next, convert the time in minutes to a time in hours.

30 min = $\frac{30}{60}$ hours = 0.5 hours

Finally, use

$$average\ speed = \frac{total\ distance\ travelled}{total\ time\ taken}$$

with the distance in kilometres and the time in hours; we get

$$average\ speed = \frac{35}{0.5} = 70 \text{ km h}^{-1}$$

ANSWERS TO EXERCISES

Exercise A

1. The volume of water in a swimming pool would be measured in the largest common measure of volume or capacity, which is the cubic metre (m^3). If it were measured in cubic centimetres (cm^3) or millilitres (ml), the numbers involved would be enormous.
2. The area of a kitchen floor would be measured in square metres (m^2) as each side of the kitchen would be measured in metres (m), again to avoid huge numbers.
3. The volume of a can of cola is commonly measured in millilitres (ml).

Exercise B

1. Using $density = \frac{mass}{volume}$

   we get a quantity in grams divided by a quantity in cm^3, so the units are g/cm^3.

2. Using $average\ speed = \frac{total\ distance\ travelled}{total\ time\ taken}$

   we get a quantity in metres divided by a quantity in seconds, so the units are m/s.

3. Using $rate\ of\ pay = \frac{pay\ received}{time\ worked}$

   we get a quantity in British pounds divided by a quantity in hours, so the units are £/h.

4. Using $average\ speed = \frac{total\ distance\ travelled}{total\ time\ taken}$

   we get a quantity in kilometres divided by a quantity in hours, so the units are km/h.

5. Using $pressure = \frac{perpendicular\ force}{area}$

   We get a quantity in Newtons divided by a quantity in m^2 , so the units are N/m^2.

Exercise C

We know that 1 $m^2 = 100^2\ cm^2 = 10\ 000\ cm^2$

so we get that 2.65 $m^2 = 2.65 \times 10\ 000\ cm^2 = 26\ 500\ cm^2$

M2. Number

M2.1

Order positive and negative integers, decimals and fractions.

Understand and use the symbols: = , ≠ , < , > , ≤ , ≥

Definitions

= is the symbol for 'is equal to' so $6 + 4 = 10$

≠ is the symbol for 'is not equal to' so $6 + 4 \neq 11$

< is the symbol for 'is less than' so $6 + 4 < 11$

> is the symbol for 'is greater than' so $6 + 4 > 9$

≤ is the symbol for 'is less than or equal to' so $6 + 5 \leq 11$ and $6 + 4 \leq 11$

≥ is the symbol for 'is greater than or equal to' so $6 + 5 \geq 11$ and $+ 7 \geq 11$

Examples: Using symbols

If $x = 6$ and $y = -4$, state whether the following are true or false:

$x + y > 4$

As $x + y = 6 + -4 = 2$ and 2 is not greater than 4 so this is false.

$x - y \neq 2$

As $x - y = 6 - -4 = 6 + 4 = 10$ and 10 is not equal to 2 so this is true.

$x + 4 \geq 9$

As $x + 4 = 6 + 4 = 10$ and 10 is greater than 9 so this is true.

$3 - y = 5$

As $3 - y = 3 - -4 = 7$ and 7 is not equal to 5 so this is false.

Ordering positive and negative integers and decimals

Integers and decimals can be ordered by considering whether they are positive or negative and then comparing the place value of each of their digits. This is usually done by writing the numbers in a column with the place value maintained.

Integers and decimals can also be ordered by comparing their placement on a

number line. The number line is usually drawn as the x- or y-axis.

On the x-axis, the larger numbers are to the right. On the y-axis, the larger numbers are higher up the axis.

Ordering fractions

Fractions can be ordered by putting them all over a common denominator and comparing the numerators or by changing them into decimals or percentages and then comparing.

Ordering a mix of integers, decimals and fractions

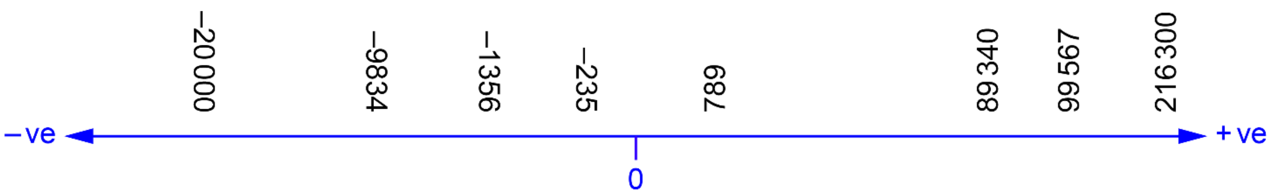
To order a mixture of number types it is usually easier to change them all into either decimals or percentages and then compare.

Example: Ordering integers

Write these integers in order of size, largest first:

89 340, 216 300, 789, -235, -1356, -20 000, 99 567, -9834

216 300, 99 567, 89 340, 789, -235, -1356, -9834, -20 000



Example: Ordering decimals

Write these decimals in order of size, largest first:

0.064, 0.00937, 0.1, 0.00876, 0.00098

0.1, 0.064, 0.00937, 0.00876, 0.00098

Example: Ordering fractions

Write these fractions in order of size, largest first:

$\frac{3}{5}, \frac{4}{10}, \frac{7}{15}, \frac{2}{3}, \frac{23}{30}$

All of the denominators divide into 30 so write all fractions as equivalent fractions with a denominator of 90:

$\frac{3}{5} = \frac{3\times6}{5\times6} = \frac{18}{30}\frac{4}{10} = \frac{4\times3}{10\times3} = \frac{12}{30}\frac{7}{15} = \frac{7\times2}{15\times2} = \frac{14}{30}$

$\frac{2}{3} = \frac{2\times10}{3\times10} = \frac{20}{30}$

$\frac{23}{30}$

Order the numerators largest first to give: 23, 20, 18, 14, 12, so the fractions in order of size largest first are:

$\frac{23}{30}, \frac{2}{3}, \frac{3}{5}, \frac{7}{15}, \frac{4}{10}$

Ordering a mixture of fractions and decimals

Write these numbers in order of size, smallest first:

$\frac{3}{7}$, 0.434, $\frac{9}{20}$, 0.0934, $\frac{89}{1000}$

Convert the fractions into decimals by dividing the numerator by the denominator to as many decimal places as are needed to determine the order of size:

$\frac{3}{7}$ = 0.428...

0.434

$\frac{9}{20}$ = 0.45

0.0934

$\frac{89}{1000}$ = 0.089

Now order the decimals as before, smallest to largest, to get 0.089, 0.0934, 0.428, 0.434, 0.45 So the list is:

$\frac{89}{1000}$, 0.0934, $\frac{3}{7}$, 0.434, $\frac{9}{20}$

M2.2

Apply the four operations (addition, subtraction, multiplication and division) to integers, decimals, simple fractions (proper and improper) and mixed numbers – any of which could be positive and negative.

Understand and use place value.

Place value

Understand and use place value for integers and decimals.

Standard place values are:

1 000 000	100 000	10 000	1000	100	10	1	•	0.1	0.01	0.001
millions	hundred thousands	ten thousands	thousands	hundreds	tens	units	decimal point	tenths	hundredths	thousandths

The table can be extended in either direction.

Each place value is a factor of 10 different from its neighbour, for example:

- Hundreds multiplied by 10 are thousands.
- Units divided by 10 are tenths.

Example: Place value

From the table [below] we can see that

The 8 in 76 890 represents 8 hundreds

The 8 in 23.986 represents 8 hundredths

The 8 in 0.008 represents 8 thousandths

1 000 000	100 000	10 000	1000	100	10	1	•	0.1	0.01	0.001
millions	hundred thousands	ten thousands	thousands	hundreds	tens	units	decimal point	tenths	hundredths	thousandths
		7	6	8	9	0	•			
					2	3	•	9	8	6
						0	•	0	0	8

Addition and subtraction of integers and decimals

To add or subtract integers or decimals, they must first be aligned in order of place value.

Example: Addition integers and decimals

Calculate 23.69 + 9.043

Referring to the table below: line up the decimal points under each other to line up numbers with the same place values. Fill in any blanks with zeros. This is not important in addition but is very important in subtraction. Add the numbers in their columns starting from the left-hand end.

9 + 4 = 13, the 1 is moved to the next column as 10 hundredths which is 1 tenth.

The rest of the numbers are added in their columns.

10	1	•	0.1	0.01	0.001
tens	units	decimal point	tenths	hundredths	thousandths
2	3	•	6	9	0
0	9	•	0	4	3
3	2	•	7	3	3

Example: Subtraction of integers and decimals

Calculate 63.79 – 9.036

Referring to the table below: line up the decimal points under each other to line up numbers with the same place values. Fill in any blanks to the left of the decimal point with zeros.

Subtract the numbers in their columns starting from the left-hand end.

6 cannot be taken from zero without introducing negative numbers so one of the hundredths is converted to 10 thousandths. 6 is then subtracted from 10 + 0 giving 4.

3 is then subtracted from 8 and 0 from 7.

9 cannot be taken directly from 3 so one of the tens is converted to 10 units and the 9 is taken from 10 + 3 giving 4. 0 is then taken from 5.

10	1	•	0.1	0.01	0.001
tens	units	decimal point	tenths	hundredths	thousandths
~~6~~ 5	10+ 3	•	7	~~9~~ 8	10+ 0
0	9	•	0	3	6
5	4	•	7	5	4

Addition and subtraction of fractions

To add or subtract fractions, each fraction must first be put over a common denominator.

To add or subtract mixed numbers, start by adding or subtracting the integer part and then add or subtract the fractional part.

Example: Addition of fractions

Find $\frac{2}{3}+\frac{3}{5}$

The lowest common multiple (LCM) of 3 and 5 is 15, we use this as our common denominator.

$\frac{2}{3}=\frac{10}{15}$ and $\frac{3}{5}=\frac{9}{15}$

So, the sum becomes:

$\frac{2}{3}+\frac{3}{5}=\frac{10}{15}+\frac{9}{15}=\frac{19}{15}=1\frac{4}{15}$

Example: Subtraction of fractions

Find $1\frac{1}{4}-\frac{7}{8}$

First we note that $1=\frac{4}{4}$ so that $1\frac{1}{4}=\frac{5}{4}$ so the subtraction becomes $\frac{5}{4}-\frac{7}{8}$

We then use the LCM of 4 and 8 to convert this to $\frac{10}{8}-\frac{7}{8}=\frac{3}{8}$

Multiplication of integers and decimals

To multiply integers or decimals, place value must be considered.

Example: Multiplication of integers

Calculate 123 × 46

Method 1 – Formal column

Write the numbers under each other in place value columns.

First multiply by the 6 units.

6 × 3 units = 18 units, 1 ten and 8 units. Move the ten to the next column.

6 × 2 tens = 12 tens. Add in the 1 moved ten to get 13 tens which is 1 hundred and 3 tens.

6 × 1 hundred = 6 hundred. Add in the 1 moved hundred to get 7 hundreds.

Now multiply by the 40. Multiply by the 10 first by putting a 0 in the units column and then by the 4. 4 × 3 = 12. Move the 1 to the next column.

4 × 2 = 8. Add in the 1 to get 9

4 × 1 = 4

Now add the 738 and the 4920 to get 5658.

Th	H	T	U
	1	2	3
	×	4	6
	$_{1}$7	$_{1}$3	8
4	$_{1}$9	2	0
5	6	5	8
$_{1}$			

Method 2 – Boxes (partitioning)

Split each number into its parts

46 = 40 + 6 and 123 = 100 + 20 + 3

Write one number across and the other number down and draw boxes across and down.

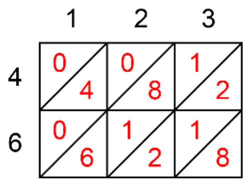
Multiply every number by every other number and put the result in the appropriate box. Now add all the results either in a long column or across and down.

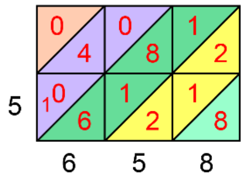
×	40	6	
100	4000	600	4600
20	800	120	920
3	120	18	138
			5658

18
120
120
600
800
4000
5658
1

Method 3 – Bones

Start in the same way as the box method but split each box in two, as shown. Now multiply the numbers which relate to the box, write each answer as 2 digits split across the box so 4 × 1 = 04 To get the answer add the numbers in the diagonal lines as shown [starting with the bottom right diagonal]





Example: Multiplication of decimals

Calculate 12.3 × 0.46

To calculate 12.3 × 0.46, first calculate 123 × 46 = 5658 [see above]

12.3 = 123 ÷ 10 and 0.46 = 46 ÷ 100 so the answer must be divided by 10 × 100 = 1000

12.3 × 0.46 = 5658 ÷ 1000 = 5.658

(Always check decimal calculations with a rough estimate, in this case 12 × 0.5 = 6)

Division of integers and decimals

The number you are dividing is called the dividend. The number you are dividing by is called the divisor. The result is called the quotient.

For example, if 360 is divided by 6 then the answer is 60. 36 is the dividend, 6 is the divisor and 60 is the quotient.

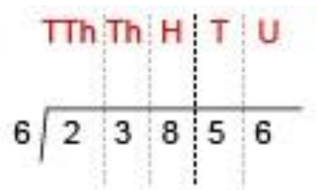
To divide by decimals, the divisor should be made into an integer by multiplying the dividend and the divisor by the same power of 10.

Example: Division of integers

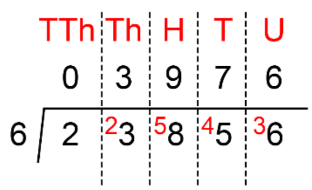
Calculate 23856 ÷ 6

Write the number you are dividing and put a ‘bus stop’ over it.

Write the number you are dividing by to the left of it.



which, on division, becomes



You can follow the explanation below using the two pictures above:

2 ÷ 6 = 0R2

Write the 0 in the ten thousand column and move the 2 to the next column to become 20 thousands giving

20 + 3 = 23 thousands

23 ÷ 6 = 3R5

Write the 3 in the thousand column and move the 5 to the next column to become 50 hundreds giving 50 + 8 = 58 hundreds

58 ÷ 6 = 9R4

Write the 9 in the hundreds column and move the 4 to the next column to become 40 tens giving 40 + 5 = 45 tens

45 ÷ 6 = 7R3

Write the 7 in the tens column and move the 3 to the next column to become 30 units giving

30 + 6 = 36 units

36 ÷ 6 = 6

so write the 6 in the units column The answer is 3976.

Example: Division of decimals

Calculate 23.856 ÷ 0.06

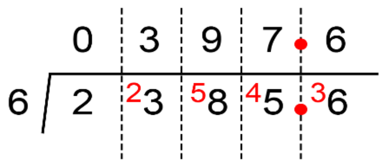
Dividing 6 by 3 is the same as dividing 60 by 30 or 6000 by 3000 or 0.006 by 0.003

You can divide or multiply both the dividend [in this cae, 23.856] and the divisor [in this case 0.06] in a division sum by the same number and leave the answer unchanged.

To divide 23.856 by 0.06 you can make the divisor [the 0.06] into an integer and to do this you multiply by 100. You must do the same to the dividend [the 23.856].

So, 23.856 ÷ 0.06 is the same as (23.856 × 100) ÷ (0.06 × 100) which is 2385.6 ÷ 6

You can then work out the answer to 2385.6 ÷ 6 by using division [as shown] to get 397.6



Pay careful attention to how the decimal point is aligned in the calculation.

You should also check the answer is right using a simple approximation: 2400 ÷ 6 = 400 which suggests the answer 397.6 obtained is the correct size.

Multiplication of fractions

To multiply two fractions, first turn any mixed numbers into improper (top-heavy) fractions, then multiply the numerators (the top numbers) and multiply the denominators (the bottom numbers).

Example: Multiplication of fractions

Calculate, giving your answer as a fraction in its lowest terms $\frac{3}{4} \times \frac{2}{3}$

Method 1: multiply fully then simplify

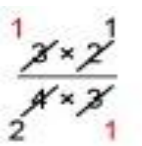
Multiply the numerators and the denominators and simplify $\frac{3}{4} \times \frac{2}{3} = \frac{3\times2}{4\times3} = \frac{6}{12} = \frac{1}{2}$

Method 2 : simplify before multiplying out

Write out as in Method 1 but simplify before multiplying the numbers:

First divide the top and bottom of $\frac{3\times2}{4\times3}$ by 3 giving $\frac{2}{4}$ and then divide both by 2 giving $\frac{1}{2}$.

This is usually shown like this:



Example: mixed fractions

Calculate: $1\frac{3}{5} \times 3\frac{3}{4}$

First convert each fraction to a top-heavy fraction:

$1\frac{3}{5} = \frac{8}{5}$ and $3\frac{3}{4} = \frac{15}{4}$

The proceed using either Method 1 or Method 2 [here cancelling before simplifying works better]:

$$1\frac{3}{5} \times 3\frac{3}{4} = \frac{8}{5} \times \frac{15}{4} = \frac{8\times15}{5\times4} = \frac{2\times3}{1\times1} = 6$$

Division of fractions

To divide fractions, invert the divisor [turn the second fraction upside down] and then multiply the result to get the answer. If there are mixed fractions in your questions, you need to convert them to top heavy fractions first

Examples: dividing fractions

Here are two examples

$15 \div \frac{3}{8} = \frac{15}{1} \div \frac{3}{8} = \frac{15}{1} \times \frac{8}{3} = \frac{15\times8}{1\times3} = \frac{5\times8}{1\times1} = 40$

$15 \div 1\frac{7}{8} = \frac{15}{1} \div \frac{15}{8} = \frac{15}{1} \times \frac{8}{15} = \frac{15\times8}{1\times15} = \frac{1\times8}{1\times1} = 8$

M2.3
Use the concepts and vocabulary of prime numbers, factors (divisors), multiples, common factors, common multiples, highest common factor, lowest common multiple, and prime factorisation (including use of product notation and the unique factorisation theorem).

Multiples

A multiple of a number lies in the times table of that number.

A common multiple for two numbers is a multiple of both. A common multiple for more than two numbers is a multiple of all the numbers.

The lowest common multiple (LCM) for two or more numbers is the smallest number that will divide by all the numbers in the question.

You should be able to:

- find multiples of a number
- find common multiples of two or more numbers
- find the lowest common multiple of two or more numbers.

Example: Finding common multiples

Find the first three common multiples of 6 and 8.

Multiples of 6 are 6, 12, 18, <u>24</u>, 30, 36, 42, <u>48</u>, 54, 60, 66, <u>72</u> …

Multiples of 8 are 8, 16, <u>24</u>, 32, 40, <u>48</u>, 56, 64, <u>72</u>, 80, 88, 96 …

Common multiples of 6 and 8 are <u>24</u>, <u>48</u>, <u>72</u> …

Factors

A factor of a number will divide into that number exactly (with no remainder). They are also known as divisors.

A common factor for two or more numbers is a number that will divide exactly into all the numbers in the question.

The highest common factor (HCF) for two or more numbers is the largest number that will divide exactly into all the numbers in the question.

You should be able to:

- find factors of a number
- find common factors of two or more numbers
- find the highest common factor of two or more numbers.

Example: Finding common factors of two or more numbers

Find all the common factors of 12 and 18

Factors of 12 are 1, 2, 3, 4, 6 and 12

Factors of 18 are 1, 2, 3, 6, 9 and 18

Common factors of 12 and 18 are 1, 2, 3 and 6

Primes

Prime numbers are numbers that have exactly two factors.

Prime factorisation involves writing a number as a product of prime numbers (as prime numbers multiplied together).
Every integer greater than 1 has a unique prime factorisation, this is known as the unique factor theorem.

You should be able to:

- decide if a number is prime
- solve problems involving prime numbers
- find the prime factorisation of a number
- write the prime factorisation of a number in index form
- use the prime factorisations of numbers to find the highest common factor and lowest common multiple
- use the prime factorisations of numbers for other purposes; e.g. to see if one number is a factor/multiple of another; or to find square roots of large numbers.

Divisibility tests are useful for helping to find factors

- a number is divisible by **2** if the last digit is even
- a number is divisible by **3** if the sum of its digits is divisible by 3
- a number is divisible by **4** if the two-digit number in the tens and units place value columns is divisible by 4
- a number is divisible by **5** if the last digit is 0 or 5
- a number is divisible by **6** if it is divisible by 2 and 3
- to find if a number is divisible by **7**, subtract 2 times the last digit from the other digits and then check if this is divisible by 7

  e.g. for 54*6*, work out 54 – *6* × 2 = 42

  42 is a multiple of 7 so 546 is a multiple of 7
- a number is divisible by **8** if the three-digit number in the hundreds, tens and units place value columns is divisible by 8
- a number is divisible by **9** if the sum of the digits is divisible by 9

Example: Deciding if a number is prime

Is 153 a prime number?

153 is divisible by 3 because the sum of the digits is divisible by 3, so it has more than two factors (at least 1, 3 and 153).

Therefore 153 is not prime.

Hint: when checking if a number is prime, check to see if it divides by prime numbers. Work through the prime numbers in increasing order.

More about primes

There are only 10 prime numbers that are less than 30, it is worth learning these:

2, 3, 5, 7, 11, 13, 17, 19, 23 and 29

Note:

- The number 1 is NOT a prime number as it only has one factor.
- The number 2 is the ONLY even prime number.
- Apart from 2 and 5, all primes end in 1, 3, 7 or 9.

Example: Writing the prime factorisation of a number in index form

Write 180 as a product of its prime factors using index notation.

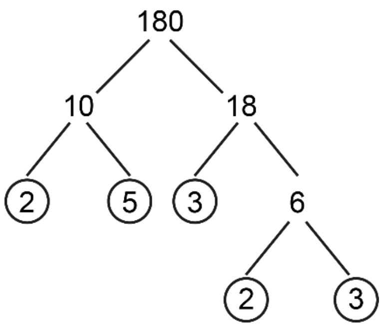
Method 1

Divisor	Answer
2	180
2	90
3	45
3	15
	5

Start by dividing by the smallest prime that will divide exactly into the number given, then the next smallest … and stop when the answer is prime.

$180 = 2 \times 2 \times 3 \times 3 \times 5 = 2^2 \times 3^2 \times 5$

Method 2



The two numbers written in the branches below a number multiply to give that number. For instance, 10 and 18 are the two numbers below 180, and 180 = 10 × 18

Circle prime numbers to show the end of a branch.

$180 = 2 \times 2 \times 3 \times 3 \times 5 = 2^2 \times 3^2 \times 5$

Example: Finding the highest common factor (HCF) of two or more numbers and finding the lowest common multiple (LCM) of two or more numbers

Find the HCF and LCM of 180 and 420.

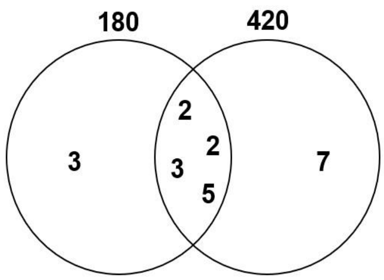
To find the HCF and LCM of 180 and 420:

First find the prime factorisation of each number:

180 = 2 × 2 × 3 × 3 × 5

420 = 2 × 2 × 3 × 5 × 7

Then write the factors in an overlapping diagram:



so that the numbers inside the 180 circle multiply to give 180 and similarly for the numbers inside the 420 circle.

To find the HCF of 180 and 420, look at the numbers in the overlap: 2 × 2 × 3 × 5 = 60.

To find the LCM of 180 and 420, multiply all the numbers in the diagram together: 2 × 2 × 3 × 3 × 5 × 7 = 1260.

How to check:

If you find the LCM and HCF of two numbers, and multiply them together, it's the same as multiplying the two numbers together.

180 × 420 = 75 600 and 60 × 1260 = 75 600

Example: Finding the highest common factor of two or more numbers and finding the lowest common multiple of two or more numbers

Find the HCF and LCM of 180, 168 and 72.

Find the HCF of 180, 168 and 72

$180 = \underline{2} \times \underline{2} \quad \times \underline{3} \times 3 \times 5$

$168 = \underline{2} \times \underline{2} \quad \times \underline{3} \qquad \times 7$

$72 \;\; = \underline{2} \times \underline{2} \times 2 \times \underline{3} \times 3$

HCF is prime factors common to all lists = $\underline{2} \times \underline{2} \times \underline{3} = 12$

Find the LCM of 180, 168 and 72

$180 = \underline{2} \times \underline{2} \quad \times \underline{3} \times 3 \times 5$

$168 = \underline{2} \times \underline{2} \quad \times \underline{3} \qquad \times 7$

$72 \;\; = \underline{2} \times \underline{2} \times 2 \times \underline{3} \times 3$

The LCM has to contain the prime factors of each of the numbers. This means that it must contain three 2s (as this is the maximum number of 2s in the prime factorisations), it must contain two 3s (as this is the maximum number of 3s in the prime factorisations) and it must contain one 5 and one 7.

LCM = $\underline{2} \times \underline{2} \times \underline{2} \times \underline{3} \times \underline{3} \times \underline{5} \times \underline{7} = 2520$

Note: this method for LCM can also be used for two numbers.

Example: Using the prime factorisations of numbers for other purposes, e.g. to see if one number is a factor of another

Show that 210 is a factor of 3780.

$210 = 2 \times 3 \times 5 \times 7$

$3780 = 2^2 \times 3^3 \times 5 \times 7$

$= 2 \times 3 \times 5 \times 7 \times 2 \times 3^2$

This shows that 210 is a factor of 3780 as the prime factors of 210 are all present in the prime factorisation of 3780.

Example: Solving problems involving prime numbers

Sarah writes down a 2-digit number.

She reverses the digits and subtracts this new number from the original number.

Can her answer ever be prime?

If the digit in the units column is x and the digit in the tens column is y, then the answer is:

$$10y + x - (10x + y) = 9y - 9x = 9(y - x)$$

If $y > x$ then this is a multiple of 9, therefore not prime.

If $y < x$ then this is a negative number therefore not prime.

Example: Using the prime factorisations of numbers for other purposes, e.g. to find square roots of large numbers

Given $129600 = 2^6 \times 3^4 \times 5^2$

Without using a calculator, use this fact to find $\sqrt{129600}$

$129600 = 2^6 \times 3^4 \times 5^2 = 2^3 \times 3^2 \times 5 \times 2^3 \times 3^2 \times 5 = (2^3 \times 3^2 \times 5) \times (2^3 \times 3^2 \times 5)$

So $\sqrt{129600} = 2^3 \times 3^2 \times 5 = 360$

M2.4

Recognise and use relationships between operations, including inverse operations.

Use cancellation to simplify calculations and expressions.

Understand and use the convention for priority of operations, including brackets, powers, roots and reciprocals.

Multiplication

Multiplication is the same as repeated addition.

Example

10 identical boxes each have a mass of 1.8 kg. What is the total mass of the 10 boxes?

You can either write down 1.8 ten times and add them all together or multiply:

1.8 × 10. 1.8 kg × 10 = 18 kg

Division

Division is the same as repeated subtraction of the same number.

Example

A container of 20 litres of lemonade is used to fill glasses, each holding 250 ml when full. How many glasses can be filled from the container?

20 l = 20 000ml

The problem can be solved either by seeing how many times 250 can be subtracted from 20 000, or by dividing 20 000 by 250.

$\frac{20000}{250} = \frac{2000}{25} = \frac{400}{5} = 80$

Indices

Indices can be used to represent repeated multiplication or division by the same number.

Example

Write 25 × 10 × 25 × 10 × 25 × 10 × 10 in index form.

Reordering we get 25 × 10 × 25 × 10 × 25 × 10 × 10 = 25 × 25 × 25 × 10 × 10 × 10 × 10

Then 25 × 25 × 25 = 253 and 10 × 10 × 10 × 10 = 10^4 So 25 × 25 × 25 × 10 × 10 × 10 × 10 = $25^3 \times 10^4$

Inverse operations

Multiplication is the inverse of division and addition is the inverse of subtraction.

Example

34 is added to a number and the result is 78. What was the number?

34 is added to a number.

The inverse of addition is subtraction.

To find the number, perform the inverse operation of addition on the 78.

? → +34 → 78

44 ← -34 ← 78

78 − 34 = 44

Algebraically, you could think of this problem as needing to find x when $34 + x = 78$ so we subtract 34 from both sides and get $x = 78 - 34 = 44$

Example

A number is divided by 8 and the result is 104. What was the number?

A number is divided by 8.

The inverse of division is multiplication.

? → ÷8 → 104

832 ← ×8 ← 104

To find the number, perform the inverse operation of division on the 104.

Algebraically, you could think of this problem as needing to find x when $\frac{x}{8} = 104$ so multiply both sides by 8 to get $x = 104 \times 8 = 832$

Cancelling

Calculations and expressions can be simplified by using cancellation. Cancellation usually means dividing the top and bottom of a fraction by the same number or algebraic term or expression.

Cancellation

Write down a calculation that is equivalent to $\frac{21\times55}{63\times85}$

We can cancel parts of the calculation to make it easier to work out.

$$\frac{21\times55}{63\times85}=\frac{21\times55}{3\times21\times85}=\frac{1\times55}{3\times85}=\frac{1\times5\times11}{3\times5\times17}=\frac{1\times11}{3\times17}$$

We cancel the 21 and 63 by dividing both by 21, we cancel the 55 and 85 by dividing both by 5.

Remember to try to simplify calculations before evaluating them.

Order of operations

The convention for the order in which operations are carried out is:

1st Brackets (also called parentheses). Either carry out the calculation in the brackets first or multiply out the bracket. If there are brackets within brackets, then start by working out the innermost bracket first.

2nd Indices (also called exponents). Carry out all index calculations including roots and reciprocals. The reciprocal of x is $\frac{1}{x}$.

3rd Division and Multiplication. These operations are equally important. Work from left to right if there is more than one of these.

4th Addition and Subtraction. These operations are equally important. Work from left to right if there is more than one of these.

This ordering can be remembered by a number of acronyms – BIDMAS and PEMDAS are two common ones.

Examples: Order of operations

Calculate: $3 + 6 \times 9$

If there are no brackets or indices then multiplication takes priority over addition so do the multiplication first and then the addition: $3 + 6 \times 9 = 3 + 54 = 57$

Calculate: $3 - (6 - 9)$

$3 - (6 - 9) = 3 - (-3)$ Evaluate the bracket first

$3 - (-3) = 3 - -3$ Remove the bracket

$3 - -3 = 3 + 3 = 6$ Remember that – – becomes +

$8 - 3(6 - 8)$ $8 - 3(6 - 8) = 8 - 3(-2)$ Evaluate the bracket first

$8 - 3(-2) = 8 - -6 = 8 + 6 = 14$ The number outside the bracket multiplies everything within the bracket

Calculate: $8 - 3(6 - 8)^2$

$8 - 3(6 - 8)^2 = 8 - 3(-2)^2$ Evaluate the bracket first

$8 - 3(-2)^2 = 8 - 3 \times 4$ Now use the index but only on the bracket and not the 3

$8 - 3 \times 4 = 8 - 12$ Next perform the multiplication $8 - 12 = -4$

Finally perform the subtraction

M2.5

Apply systematic listing strategies. (For instance, if there are m ways of doing one task and for each of these tasks there are n ways of doing another task, then the total number of ways the two tasks can be done in order is $m \times n$ ways.)

If there are m ways of doing one task and for each of these, there are n ways of doing another task, then the total number of ways the two tasks can be done in order is $m \times n$ ways.

Examples:

Systematic listing for codes

A code has 4 digits.

Each digit is between 0 and 9 inclusive.

How many different codes are possible?

10 × 10 × 10 × 10 = 10000

Note: This is not surprising as there are 10 000 different numbers between 0000 and 9999.

How many possible codes are there where each digit is different?

10 × 9 × 8 × 7 = 5040

There are 10 choices for the first digit. There are then 9 digits that are different to the first digit that could be used for the second digit. There are then 8 digits that are different to the first digit, and the second digit that could be used for the third digit. Finally, there are 7 digits that are different to the first, second and third digits that could be used for the fourth digit.

How many possible codes are there where there are two digits the same next to one another and the other two digits are unique?

There are three possible positions for the repeated digit:

1st and 2nd digits OR 2nd and 3rd digits OR 3rd and 4th digits

We need to consider how many ways there are of choosing a digit to be repeated and two other digits:

10 × 9 × 8

However, the repeated number could be in one of three positions, so the number of possible codes is:

3 × (10 × 9 × 8) = 2160

How many possible codes are there where there are at least two digits the same?

The number of codes with two or more digits repeated can be calculated by starting with the total number of possible codes and subtracting the number of codes where each digit is unique:

10000 − 5040 = 4960

Example

Martha has 4 jumpers, 5 pairs of trousers and 3 pairs of trainers.

How many different combinations of outfit can Martha make?

$4 \times 5 \times 3 = 60$

Example

A new licence plate system is being introduced.

The licence plate will have two letters followed by up to 4 digits from 0 to 9.

How many licence plates can be created with two letters followed by exactly 4 digits?

There are 26 possibilities for each letter (assuming that all letters are allowable).

There are 10 possibilities for each digit (0 to 9).

$26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6760000$

How many licence plates can be created with two letters followed by exactly 4 digits, if the two letters cannot be the same?

If the two letters cannot be the same, there are 26 possible letters for the first letter and 25 possible letters for the second letter.

$26 \times 25 \times 10 \times 10 \times 10 \times 10 = 6500000$

How many licence plates can be created with two letters followed by exactly 4 digits, if the two letters cannot be the same and no digit is repeated?

If the two letters cannot be the same, there are 26 possible letters for the first letter and 25 possible letters for the second letter.

If the four digits cannot be repeated, there are 10 possibilities for the first digit, 9 possibilities for the second digit, 8 possibilities for the third digit and 7 possibilities for the fourth digit.

$26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3276000$

How many licence plates can be created with two letters and up to 4 digits? (There must be at least 1 digit, and treat 01 as different to 1 in the context of the licence plate).

We can consider the number of possibilities with one digit, then with two digits, then with three digits, and finally with four digits.

One digit

$26 \times 26 \times 10 = 6760$

Two digits

26 × 26 × 10 × 10 = 67600

Three digits

26 × 26 × 10 × 10 × 10 = 676000

Four digits

26 × 26 × 10 × 10 × 10 × 10 = 6760000

Total number of possibilities:

6760000 + 676000 + 67600 + 6760 = 7510360

M2.6
Use and understand the terms: *square*, *positive* and *negative square root*, *cube* and *cube root*.

Squares

- The square of any number is found by multiplying the number by itself. For example, the square of 2.5 is 2.5^2, which is 2.5 × 2.5 which is equal to 6.25

- The square of either a positive or a negative number is a positive number. For example, $(-4)^2 = -4 \times -4 = 16$

- In general, the square of x is x^2.

Square roots

- The square root of a number is the **positive** number that when multiplied by itself gives the original number.

- The square root of 9 is written as $\sqrt{-9}$. So $\sqrt{-9} = 3$.

- The square root of a number is, by definition, positive. However, we can talk about a 'negative square root' to mean the negative number that when multiplied by itself gives the original number. So, the square root of 9 is 3 only, but we can say that the 'negative square root' of 9 is −3.

Cube numbers

- The cube of any number is found by multiplying a number by itself, then multiplying by this number again. For example, the cube of 1.2 is $1.2^3 = 1.2 \times 1.2 \times 1.2 = 1.728$

- The cube of a positive number is a positive number, and the cube of a negative number is a negative number.

- In general, the cube of x is x^3.

Cube roots

The cube root of a number is a number that when multiplied by itself and multiplied by itself again gives the original number. The cube root of a positive number is positive, and the cube root of a negative number is negative. For example, 4 × 4 × 4 = 64, so the cube root of 64 is 4.

M2.7

Use index laws to simplify numerical expressions, and for multiplication and division of integer, fractional and negative powers.

Index numbers or powers

The power a number is raised to is the index (plural: indices).

For example,

$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$ so 2^5 is 32 written in index form.

You should be able to:

- Convert numbers into index form.
- Evaluate numbers written in index form.

Index laws

For powers of the same number (base):

Multiplication

To multiply powers of the same number, add the indices.

$a^m \times a^n = a^{m+n}$

Note also: $(ab)^n = a^n b^n$

You should be able to:

- Multiply powers of the same base.

Division

To divide powers of the same number, subtract the indices.

$$a^m \div a^n = a^{m-n}$$

You should be able to:

- Divide powers of the same base.

Also:

- Any number raised to the power 0 is equal to 1.
- In general, $a^0 = 1$, for all non-zero values of a.
- Any number to the power of 1 is just the number itself. In general, $a^1 = a$

You should be able to:

- Evaluate and use a number raised to the power ZERO.
- Evaluate 1 raised to any power.
- Write and use any number raised to the power 1.

Fractions

The power applies to both the numerator and the denominator.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

You should be able to:

- Evaluate a fraction raised to a power.

Negative powers

A number raised to a negative power can be written as 1 over the number to the positive power.

$$a^{-m} = \frac{1}{a^m}$$

You should be able to:

- Evaluate and use any number raised to a negative power.

Raising powers to a further power

To raise a power to a further power, multiply the powers.

$(a^m)^n = a^{mn}$

You should be able to:

- Evaluate a number in index form raised to a further power.

Fractional powers

- The power $\frac{1}{2}$ is the same as the square root.
- The power $\frac{1}{3}$ is the same as the cube root.
- The power $\frac{1}{4}$ is the same as the fourth root etc.

$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

You should be able to:

- Evaluate a number with a fractional power.
- Evaluate a number with a fractional power where the numerator is greater than 1.

Examples using indices

Write 243 as a power of 3.

$243 = 3 \times 3 \times 3 \times 3 \times 3 = 3^5$

Evaluate 2^7.

$2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$

Write as a single power of 5

$5^3 \times 5^4$

Remember to add the powers

$5^{3+4} = 5^7$

Write as a single power of 2

$2^8 \div 2^5$

Remember to subtract the powers

$2^{8-5} = 2^3$

Evaluate $(1.2)^0 + 5^3 + 1^5 - 7^1$

$(1.2)^0 + 5^3 + 1^5 - 7^1 = 1 + 125 + 1 - 7 = 120$

Work out $\left(\frac{3}{4}\right)^4$

Remember to raise both the numerator and the denominator of the fraction to the power outside the brackets.

$$\left(\frac{3}{4}\right)^4 = \frac{3^4}{4^4} = \frac{81}{256}$$

Evaluate 5^{-3}

$$5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

***Work out* $(2^3)^{-2}$**

Remember to multiply the powers:

$(2^3)^{-2} = 2^{3\times -2} = 2^{-6} = \frac{1}{2^6} = \frac{1}{64}$

Evaluate $49^{-\frac{1}{2}}$

$49^{-\frac{1}{2}} = \frac{1}{49^{\frac{1}{2}}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

Evaluate $16^{-\frac{3}{4}}$

$16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}} = \frac{1}{\left(\sqrt[4]{16}\right)^3} = \frac{1}{2^3} = \frac{1}{8}$

M2.8

Interpret, order and calculate with numbers written in standard index form (standard form); numbers are written in standard form as $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

Standard form

A number in standard form is of the form $a \times 10^n$ where $1 \leq a < 10$ and n is a positive or negative integer.

Examples

Express 124 in standard form

To put 124 into the form $a \times 10^n$, first decide where the decimal point goes in 124 to give a number between 1 and 10

$a = 1.24$

Now consider the relationship between 124 and 1.24.

$1.24 = 124 \div 100 = 124 \div 10^2$

If $1.24 = 124 \div 10^2$ then $124 = 1.24 \times 10^2$

Express 124000 in standard form

To put 124000 into the form $a \times 10^n$, first decide where the decimal point goes in 124 to give a number between 1 and 10

Again $a = 1.24$

$1.24 = 124000 \div 100000 = 124 \div 10^5$

If $1.24 = 124 \div 5$ then $124000 = 1.24 \times 10^5$

Express 0.124 in standard form

To put 0.124 into the form $a \times 10^n$, first decide where the decimal point goes in 124 to give a number between 1 and 10

Again $a = 1.24$

$1.24 = 0.124 \times 10 = 0.124 \times 10^1$

If $1.24 = 0.124 \times 10^1$ then $0.124 = 1.24 \div 10^1 = 1.24 \times 10^{-1}$

Remember that multiplying by $\frac{1}{10^x}$ is the same as dividing by 10^x which is the same as multiplying by 10^{-x}.

Express 0.0000124 in standard form

To put 0.0000124 into the form $a \times 10^n$, first decide where the decimal point goes in 124 to give a number between 1 and 10

Again $a = 1.24$

$1.24 = 0.0000124 \times 100000 = 0.0000124 \times 10^5$

If $1.24 = 0.0000124 \times 10^5$ then $0.0000124 = 1.24 \div 10^5 = 1.24 \times 10^{-5}$

Ordering numbers in standard form

Numbers in standard form, $a \times 10^n$, can be placed in order of size by first looking at n and then at a.

Example: Ordering numbers in standard form

Place the following in order of size, smallest first:

6.1×10^4

3.6×10^7

8.135×10^{-2}

9.6×10^{-4}

6×10^4

The numbers given are all in standard form so first place the numbers in order of the indices of the 10s and then look at the numbers with the same indices:

The indices of 10, in order, are -4,-2,4,4 and 6.

The two numbers multiplied by $10^4$are 6.1 and 6 of which 6 is the smaller.

The list in order of size, smallest first, is:

9.6×10^{-4}, 8.135×10^{-2}, 6×10^4, 6.1×10^4, 3.6×10^7

The indices are in order from smallest to largest and the two numbers with the same index power of 10 are also in order, smallest first.

Calculating with numbers in standard form

Calculations involving numbers in standard form follow the normal rules of calculation with index numbers.

Example: calculating with numbers in standard form

Given p = 4×10^5 and q = 8×10^{-3} we can calculate various combinations as follows:

pq To multiply numbers in standard form, multiply the numbers and then multiply the powers of 10 by adding the indices.

$4 \times 10^5 \times 8 \times 10^{-3} = (4 \times 8) \times (10^5 \times 10^{-3}) - 32 \times 10^2$

32×10^2 is not in standard form as 32 is not between 1 and 10.

In standard form, $32 = 3.2 \times 10^1$ so $32 \times 10^2 = 3.2 \times 10^1 \times 10^2 = 3.2 \times 10^3$

p/q To divide numbers in standard form, divide the numbers and then divide the powers of 10 by subtracting the indices.

$(4 \times 10^5) \div (8 \times 10^{-3}) = (4 \div 8) \times (10^5 \div 10^{-3}) = 0.5 \times 10^{5--3} = 0.5 \times 10^8$

0.5×10^8 is not in standard form as 0.5 is not between 1 and 10.

$0.5 = 5 \times 10^{-1}$

$0.5 \times 10^8 = 5 \times 10^{-1} \times 10^8 = 5 \times 10^7$

q^2

$q^2 = (8 \times 10^{-3})^2 = (8 \times 10^{-3}) \times (8 \times 10^{-3}) = (8 \times 8) \times (10^{-3} \times 10^{-3}) = 64 \times 10^{-6}$

64×10^{-6} is not in standard form as 64 is not between 1 and 10.

In standard form, $64 = 6.4 \times 10^1$ so $64 \times 10^{-6} = 6.4 \times 10^1 \times 10^{-5} = 6.4 \times 10^{-5}$

p+q

$6 \times 10^4 + 8 \times 10^3$

Method 1

To add (or subtract) numbers in standard form, take the numbers out of standard form first:

$6 \times 10^4 + 8 \times 10^3 = 60000 + 8000 = 68000$

$68000 = 6.8 \times 10^4$

Method 2

Take out the common factors of 10 first:

$6 \times 10^4 + 8 \times 10^3 = 10^3(6 \times 10^1 + 8) = 68 \times 10^3 = 6.8 \times 10^4$

M2.9

Convert between terminating decimals, percentages and fractions.

Convert between recurring decimals and their corresponding fractions.

Definitions

Fractions, decimals and percentages all describe a proportion of a group, number or whole.

A terminating decimal is a decimal which has a finite number of digits.

A recurring decimal is a decimal which has repeating digits or a repeating pattern of digits. A recurring decimal has an infinite number of digits.

Converting between forms

[see the examples below too]

- To convert a mixed number to an improper fraction, convert the integer part into a fraction with the same denominator as the fractional part then add this onto the fractional part.
- To convert an improper fraction into a mixed number, divide the numerator by the denominator. The answer to the division is the whole number part and any remainder is written over the original denominator as the fractional part.
- To cancel a fraction to its lowest terms, divide both the numerator and the denominator by the highest common factor.
- Equivalent fractions can be found by multiplying or dividing the numerator and the denominator of the fraction by the same value.
- To convert a fraction to a decimal, divide the numerator by the denominator of the fraction. Alternatively, use equivalent fractions to write the fraction over a denominator which is a power of 10 and then write this as a decimal by using place value.
- To convert a decimal to a percentage, multiply by 100.
- To convert a percentage to a decimal, divide by 100.
- To convert a terminating decimal to a fraction, use the smallest place value to identify the denominator and write the digits in the decimal as the numerator. For example, 0.0307 = $\frac{307}{10000}$
- since the 7 in the decimal is in the 10 000ths place value column.
- To convert a recurring decimal to a fraction: see examples below

You should be able to:

- Convert between mixed numbers and improper fractions.
- Find equivalent fractions.
- Cancel fractions to their lowest terms.
- Convert a fraction to a decimal.
- Convert between decimals and percentages.
- Convert a terminating decimal to a fraction.
- Convert a fraction to a recurring decimal.
- Convert a recurring decimal to a fraction.
- Order fractions, decimals and percentages by converting them to equivalent forms.

Note: A terminating decimal comes from a fraction where the only prime factors in the denominator of the simplified fraction are 2 and/or 5. If there are prime factors other than 2 and/or 5 in the denominator of the fraction, the decimal will be recurring.

Examples:

Converting between mixed numbers and improper fractions

Write $4\frac{5}{12}$ as an improper fraction.

$$4\frac{5}{12} = 4 + \frac{5}{12} = \frac{4\times12}{12} + \frac{5}{12} = \frac{48}{12} + \frac{5}{12} = \frac{53}{12}$$

Write $\frac{17}{5}$ as a mixed number.

Divide the numerator by the denominator.

17 ÷ 5 = 3 r 2

This gives an answer of $3\frac{2}{5}$

Cancelling fractions to their lowest terms

Cancel $\frac{630}{1638}$ to its lowest terms.

Method 1

You can rewrite the numerator and denominator as a product of primes to help cancelling.

$$\frac{630}{1638} = \frac{\cancel{2}\times\cancel{3}\times\cancel{3}\times 5\times\cancel{7}}{\cancel{2}\times\cancel{3}\times\cancel{3}\times\cancel{7}\times 13}$$

$= \frac{5}{13}$

Method 2

Repeatedly cancel by 2, 3 etc.

$$\frac{630}{1638} \overset{\div 2}{=} \frac{315}{819} \overset{\div 3}{=} \frac{105}{273} \overset{\div 3}{=} \frac{35}{91} \overset{\div 7}{=} \frac{5}{13}$$

Finding equivalent fractions

Find the value of x where

$\frac{105}{168} = \frac{x}{120}$

We can either work to convert 168 to 120:

$$\frac{105}{168} \overset{\div 7}{=} \frac{15}{24} \overset{\times 5}{=} \frac{75}{120}$$

giving $x = 75$

or we can "cross multiply":

$105 \times 120 = 168\,x$

And then rearrange and cancel:

$x = \frac{105\times120}{168} = 75$

Converting a fraction to a decimal

Convert $\frac{7}{20}$ and $\frac{5}{8}$ to decimals.

For $\frac{7}{20}$ use equivalent fractions to convert to a fraction with a denominator of 10^n

$$\frac{7}{20} = \frac{35}{100} = 0.35$$

For $\frac{5}{8}$ use short division

$$\begin{array}{r} 0.6\,2\,5 \\ 8\overline{)\,5.{}^{5}0{}^{2}0{}^{4}0} \end{array}$$

Converting between decimals and percentages

Convert $35\frac{1}{5}\%$ to a decimal.

$$35\frac{1}{5}\% = 35\frac{2}{10}\% = \frac{35.2}{100} = 0.352$$

Convert 5.5 to a percentage.

$(5.5 \times 100)\% = 550\%$

Converting a terminating decimal to a fraction

Convert 0.725 to a fraction in its lowest terms.

The 5 in 0.725 is in the 1000ths place value column.

Write the digits of the decimal as the numerator and use 1000 as the denominator.

$$\frac{725}{1000} \overset{\div 5}{=} \frac{145}{200} \overset{\div 5}{=} \frac{29}{40}$$

Converting a fraction to a recurring decimal

Convert $\frac{3}{11}$ to a recurring decimal.

Convert it to a fraction out of 99, so $\frac{3}{11} = \frac{27}{99} = 0.\dot{2}\dot{7}$

Note that the two dots show that "27" recurs: 0.27272727272727...

Convert $\frac{47}{90}$ to a recurring decimal.

To make the calculation easier, divide the numerator and denominator by 10 so $\frac{47}{90} = \frac{4.7}{9}$

Then use a method of division

$$\begin{array}{r} 0.5\,2\,2\,.... \\ 9\overline{)\,4.{}^{4}7{}^{2}0{}^{2}0} \end{array}$$

$= \ 0.522222\ ... \ = \ 0.5\dot{2}$

The 2 is recurring because each time after the first 2 we will always be dividing 9 into 20 to get 18 remainder 2, getting the same result.

Converting a recurring decimal to a fraction

Convert $0.\dot{4}\dot{2}$ to a fraction in its lowest terms.

Call the decimal d, so that d = $0.\dot{4}\dot{2}$

Find $10^2 \times$ d (2 is chosen such that when *d* is subtracted from this result the recurring part of the decimal is eliminated; essentially, there are two digits recurring – the 4 and the 2).

100d = $42.\dot{4}\dot{2}$

d = $0.\dot{4}\dot{2}$

Subtract these two results to eliminate the recurring part

99d = 42

So we rearrange to get d $= \frac{42}{99} = \frac{14}{33}$

Convert 0.1322222… = $0.13\dot{2}$ to a fraction in its lowest terms.

Call the decimal d, so that d = $0.13\dot{2}$

Find 10 × d

10d = $1.3222222 \ldots = 1.3\dot{2}$

Subtract these two results to eliminate the recurring part to give

9d = 1.19

d $= \frac{1.19}{9}$

Then multiply the numerator and denominator by 100 to ensure there are no decimals in the fraction

$\text{d} = \frac{119}{900}$

Ordering fractions, decimals and percentages by converting them to equivalent forms

Write these numbers in ascending order:

$\frac{4}{11}$, 37%, $\frac{7}{20}$, $0.3\dot{6}$

Write all four of the numbers in the same form so that they can be compared. Converting to decimals is often the most suitable approach.

$\frac{4}{11} = \frac{36}{99} = 0.\dot{3}\dot{6} = 0.3636\ldots$

37% = 0.37 = 0.370

$\frac{7}{20} = \frac{35}{100} = 0.350$

$0.3\dot{6} = 0.366\ldots$

Writing these in ascending order, we have

$\frac{7}{20}$, $\frac{4}{11}$, $0.3\dot{6}$, 37%

M2.10

Use fractions, decimals and percentages interchangeably in calculations.

Understand equivalent fractions.

Use fractions, decimals and percentages interchangeably in calculations

Many problems involve numbers being given in different forms. In those situations, you have to choose the most appropriate method of calculation: whether to use fractions, decimals or percentages.

When multiplying a decimal by a fraction, either change both into fractions – usually the easier option – or change both into decimals.

Equivalent fractions

To find fractions equivalent to a given fraction, either multiply numerator and denominator by the same number or divide numerator and denominator by the same number:

$$\frac{x}{y} = \frac{nx}{ny} = \frac{\left(\frac{x}{n}\right)}{\left(\frac{y}{n}\right)}$$

Fractions, decimals and percentages in calculations

The sale price of a chair is $\frac{3}{4}$ of the price of the chair before the sale. On the final day of the sale, the price is reduced to 0.6 of the sale price.

What percentage of the price before the sale is the final day price?

Let the original price be x.

The sale price is $\frac{3}{4}$ of this so the sale price is $\frac{3x}{4}$

The next part of the information is given as a decimal so to use it with the fraction $\frac{3x}{4}$ it would be easier to change 0.6 into a fraction:

0.6 is $\frac{3}{5}$

So 0.6 of the sale price is $\frac{3}{5} \times \frac{3x}{4} = \frac{9x}{20}$

The sale price is $\frac{9}{20}$ of the original price.

This, as a percentage, $\frac{9}{20} \times 100 = \frac{900}{20} = 45\ \%$

Fractions, decimals and percentages in calculations

The pupils in school year 8 are asked to name the way they come to school most often.

52% of Year 8 use the school bus service, $\frac{1}{5}$ of Year 8 walk to school and 0.1 of Year 8 cycle to school. The rest of the pupils in Year 8 come to school by car. What fraction of Year 8 come to school by car?

Method 1

Working in fractions:

$52\% = \frac{52}{100} = \frac{52 \div 4}{100 \div 4} = \frac{13}{25}$

0.1 is $\frac{1}{10}$

Add $\frac{13}{25} + \frac{1}{5} + \frac{1}{10} = \frac{26+10+5}{50} = \frac{41}{50}$

So the fraction who come by car is $1 - \frac{41}{50} = \frac{9}{50}$

Method 2

Working in percentages:

$\frac{1}{5}$ = 20%

0.1 = 10%

Add the percentages: 52% + 10% + 20% = 82%

The percentage who come by car is 100% − 82% = 18%

As a fraction 18% is $\frac{18}{100} = \frac{9}{50}$

Method 3

In decimals:

52% = 0.52

$\frac{1}{5}$ = 0.2

Adding 0.52 + 0.2 + 0.1 = 0.82

The decimal of Year 8 who come by car is 1 − 0.82 = 0.18

As a fraction 0.18 = $\frac{18}{100} = \frac{9}{50}$

M2.11

Calculate exactly with fractions, surds and multiples of π.

Simplify surd expressions involving squares, e.g. $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4}\,\sqrt{3} = 2\sqrt{3}$, and rationalise denominators; for example, candidates could be asked to rationalise expressions such as:

$\frac{3}{\sqrt{7}}$, $\frac{5}{3+2\sqrt{5}}$, $\frac{7}{2-\sqrt{3}}$, $\frac{3}{\sqrt{5}-\sqrt{2}}$

Fractions

When calculating exactly with fractions, it is usual to leave the answer as a fraction unless the question asks for decimals or percentages.

Calculate with surds

A surd is an expression that includes square roots or cube roots etc., where the square root (or cube root etc.) cannot be simplified to an integer or fraction.

Here are some examples of surds: $\sqrt{2}$, $1 + 2\sqrt{3}$, $3\sqrt{7}$

Surds can be multiplied and divided so, $\sqrt{x}\,\sqrt{y} = \sqrt{xy}$ and $\sqrt{x} \div \sqrt{y} = \sqrt{\frac{x}{y}}$

Surds **cannot** be simply added and subtracted so:

$$\sqrt{x} + \sqrt{y} \neq \sqrt{x + y}$$

$$\sqrt{x} - \sqrt{y} \neq \sqrt{x - y}$$

Multiples of π

Calculating exactly with multiples of π means leaving the answer in terms of π and not using approximations such as π ≈ 3

Simplifying surds

Surds can be simplified by splitting and simplifying so $\sqrt{12} = \sqrt{4} \times \sqrt{3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

Rationalising the denominator

To rationalise the denominator of a fraction means to write the expression without surds in the denominator.

If the denominator is a single surd then, to rationalise the denominator, multiply numerator and denominator by the surd so:

$$\frac{3}{\sqrt{7}} = \frac{3 \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{3\sqrt{7}}{7}$$

If the denominator is of the form $x + \sqrt{y}$ then multiply numerator and denominator by $x - \sqrt{y}$ so the denominator is a difference of two squares $(x + \sqrt{y})(x - \sqrt{y}) = x^2 - y$

$$\frac{5}{3 + 2\sqrt{5}} = \frac{5(3 - 2\sqrt{5})}{(3 + 2\sqrt{5})(3 - 2\sqrt{5})} = \frac{5(3 - 2\sqrt{5})}{3^2 - (2\sqrt{5})^2} = \frac{5(3 - 2\sqrt{5})}{9 - 20} = \frac{5(3 - 2\sqrt{5})}{-11} = \frac{5(2\sqrt{5} - 3)}{11}$$

If the denominator is of the form $x - \sqrt{y}$ then multiply numerator and denominator by $x + \sqrt{y}$ so the denominator is a difference of two squares $(x - \sqrt{y})(x + \sqrt{y}) = x^2 - y$

$$\frac{5}{3 - 2\sqrt{5}} = \frac{5(3 + 2\sqrt{5})}{(3 - 2\sqrt{5})(3 + 2\sqrt{5})} = \frac{5(3 + 2\sqrt{5})}{3^2 - (2\sqrt{5})^2} = \frac{5(3 + 2\sqrt{5})}{9 - 20} = \frac{5(3 + 2\sqrt{5})}{-11} = \frac{-5(3 + 2\sqrt{5})}{11}$$

If the denominator is of the form $\sqrt{x} - \sqrt{y}$ then multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$ so the denominator is a difference of two squares $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = x - y$

$$\frac{3}{\sqrt{5} - \sqrt{2}} = \frac{3(\sqrt{5} + \sqrt{2})}{(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})} = \frac{3(\sqrt{5} + \sqrt{2})}{(\sqrt{5})^2 - (\sqrt{2})^2} = \frac{3(\sqrt{5} + \sqrt{2})}{5 - 2} = \frac{3(\sqrt{5} + \sqrt{2})}{3} = \sqrt{5} + \sqrt{2}$$

Examples

Calculating with fractions

Calculate exactly: $\frac{2}{3}+\frac{3}{5}-\frac{7}{12}$

Put the fractions over a common denominator – in this case the LCM of 3,5 and 12 is 60 but you could use 3 × 5 × 12 = 180

$\frac{2}{3}+\frac{3}{5}-\frac{7}{12}=\frac{40}{60}+\frac{36}{60}-\frac{35}{60}=\frac{40+36-35}{60}=\frac{41}{60}$

Calculating with surds

Calculate exactly: $3\sqrt{3}\times 7\sqrt{12}$

$3\sqrt{3}\times 7\sqrt{12}=3\sqrt{3}\times 7\sqrt{3}\sqrt{4}=3\sqrt{3}\times 7\sqrt{3}\times 2=42\left(\sqrt{3}\right)^2=42\times 3=126$

Calculating with multiples of π

The radius of a circle is 4 cm. What is the exact value of the area of the circle in cm^2?

Using area of a circle is $\pi r^2 = \pi \times 4^2 cm^2 = 16\pi\ cm^2$

This is the exact value; do not substitute approximations for π but leave the answer in terms of π.

Simplifying surds

$\sqrt{128}=\sqrt{2}\sqrt{64}=\sqrt{2}\times 8=8\sqrt{2}$

$\sqrt{21}\times\sqrt{28}=\sqrt{3\times 7}\times\sqrt{4\times 7}=\sqrt{3}\sqrt{7}\sqrt{4}\sqrt{7}=\sqrt{3}\times 2\times\left(\sqrt{7}\right)^2=14\sqrt{3}$

Rationalising the denominator

Rationalise the denominator and simplify:

$\frac{1}{\sqrt{2}}=\frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{2}}{\left(\sqrt{2}\right)^2}=\frac{\sqrt{2}}{2}$

$\frac{2+\sqrt{3}}{2-\sqrt{3}}$

$=\frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$ multiply top and bottom by $\left(2+\sqrt{3}\right)$

$= \frac{(2+\sqrt{3})^2}{4-3}$ because the bottom [denominator can be calculated using the difference of squares

$= \frac{4+4\sqrt{3}+3}{1}$ multiplying out the numerator

$= 7 + 4\sqrt{3}$

$\frac{\sqrt{5}}{\sqrt{15}\,(\sqrt{7}-\sqrt{2})}$

$\frac{\sqrt{5}}{\sqrt{15}\,(\sqrt{7}-\sqrt{2})} = \frac{\sqrt{5}}{\sqrt{15}\,(\sqrt{7}-\sqrt{2})} \times \frac{\sqrt{15}\,(\sqrt{7}+\sqrt{2})}{\sqrt{15}\,(\sqrt{7}+\sqrt{2})}$ multiply to rationalise the denominator

Then simplify in stages [we are writing out more stages than is usual to help you follow]:

$\frac{\sqrt{5}\sqrt{15}(\sqrt{7}+\sqrt{2})}{15\,(7-2)} = \frac{\sqrt{75}(\sqrt{7}+\sqrt{2})}{15\,(7-2)} = \frac{\sqrt{3}\,\sqrt{25}(\sqrt{7}+\sqrt{2})}{15\times 5} = \frac{\sqrt{3}\times\cancel{5}\times(\sqrt{7}+\sqrt{2})}{15\times\cancel{5}} = \frac{\sqrt{3}\,(\sqrt{7}+\sqrt{2})}{15}$

M2.12

Calculate with upper and lower bounds, and use in contextual problems.

Finding upper and lower bounds

If a number is rounded to a value, x, then the greatest lower bound is the smallest number which would round up to x. The least upper bound is the smallest number which would round up to a number bigger than x. [Instead of referring to greatest lower bound and least upper bounds, we shall just a call them the lower bound and the upper bound or GLB and LUB]

If the number were 3.84 correct to 2 decimal places, then:

the lower bound would be 3.835 which is the smallest number which would round to 3.84

the upper bound would be 3.845 which is the smallest number which rounds to a number bigger than 3.84.

Calculating with bounds using multiplication and addition only

To find the upper bound of a calculation involving only multiplication and addition, use the upper bounds of the numbers or quantities involved.

LUB (A×B) = LUB(A) × LUB(A)

LUB (A+B) = LUB(A) + LUB(A)

To find the lower bound of a calculation involving only multiplication and addition, use the lower bounds of the numbers or quantities involved.

GLB (A×B) = GLB(A) × GLB(A)

GLB (A+B) = GLB(A) + GLB(A)

Calculating with bounds and division

To find the least upper bound of a calculation involving division, use the upper bound of the dividend and the lower bound of the divisor.

$$\text{LUB}\left(\frac{A}{B}\right) = \frac{\text{LUB (A)}}{\text{GLB(B)}}$$

To find the greatest lower bound of a calculation involving division, use the lower bound of the dividend and the upper bound of the divisor.

$$\text{GLB}\left(\frac{A}{B}\right) = \frac{\text{GLB (A)}}{\text{LUB(B)}}$$

Calculating with bounds and subtraction

To find the upper bound of a calculation involving subtraction, subtract the lower bound from the upper bound.

LUB (A-B) = LUB(A) - GLB(A)

To find the lower bound of a calculation involving subtraction, subtract the upper bound from the lower bound.

GLB (A-B) = GLB(A) - LUB(A)

Examples

Finding upper and lower bounds

The numbers and quantities given have their level of accuracy shown. Find their upper and lower bounds.

236 grams is a quantity correct to the nearest gram.

The smallest number which when written correct to the nearest gram is 236 g, is 235.5 g – the lower bound.

The number above 236 correct to the nearest gram is 237 g. The smallest number which rounds to this when rounded to the nearest gram is 236.5 g – the upper bound.

or

The bounds are 236 ± 0.5 g so 235.5 g and 236.5 g.

Calculating with bounds using multiplication and addition only

The length of a rectangle is 23.6 cm correct to one decimal place, and its width is 14.7 cm correct to one decimal place. What is the upper bound of:

the perimeter of the rectangle?

the area of the rectangle?

The upper bound of the length is 23.65 cm and the upper bound of the width is 14.75 cm.

The upper bound of the perimeter is 2 × 23.65 + 2 × 14.75 = 76.8 cm

The upper bound of the area is 23.65 × 14.75 = 348.8375 cm^2

Calculating with bounds and division

If the volume of a piece of wood is 270 cm^3, correct to the nearest 10 cm^3, and the mass of the wood is 540 g correct to the nearest 10 g, what are the upper and lower bounds of the density of the wood in g/cm^3 correct to 2 d.p.?

The upper and lower bounds of 270 are 265 and 275.

The upper and lower bounds of 540 are 535 and 545.

density = $\frac{\text{mass}}{\text{volume}}$

The upper bound of the density is $\frac{275}{535} = 0.51$ g/cm^3

The lower bound of the density is $\frac{265}{545} = 0.49$ g/cm^3

Calculating with bounds and subtraction

If a, b, and c are given, correct to the nearest whole number, as a = 6, b = 3 and c = 2 what are the upper and lower bounds of $b^2 - 4ac$?

The upper bound of 6 is 6.5 and the lower bound is 5.5

The upper bound of 3 is 3.5 and the lower bound is 2.5

The upper bound of 2 is 2.5 and the lower bound is 1.5

For the upper bound, the largest possible value of b^2 and the smallest value of 4ac are needed:

the upper bound is $6.5^2 - 4 \times 2.5 \times 1.5 = 27.25$

For the lower bound, the smallest possible value of b^2 and the largest value of 4ac are needed:

the lower bound is $5.5^2 - 4 \times 3.5 \times 2.5 = -4.75$

M2.13

Round numbers and measures to an appropriate degree of accuracy, e.g. to a specified number of decimal places or significant figures.

Use inequality notation to specify simple error intervals due to truncation or rounding.

Rounding numbers to a given number of decimal places (d.p)

To round to, for example, 3 d.p.:

if the number in the 4th decimal place is 5 or more add 1 to the number in the third decimal place otherwise leave it unchanged.

Rounding numbers to a given number of significant figures (sig. figs.)

To round to, for example, 3 sig. figs.: count the three significant figures from left to right starting with the non-zero digit furthest to the left cut off all digits to the right of the third significant figure replacing them with zeros if they are to the left of the decimal point if the 4th significant figure is 5 or more add 1 to the third significant figure otherwise leave it unchanged.

Rounding measures to specified accuracy

Rounding measures correctly means knowing the conversion between standard units; for example, rounding metres to the nearest kilometre means dividing the metres by 1000 first, to convert to kilometres, and then rounding to the nearest whole number.

Truncation of a decimal

Truncating a decimal to, for example, one decimal place means cutting off all numbers to the right of the first decimal place.

Rounding numbers to a given number of decimal places (d.p.)

Round 31.56387 to 4 d.p

Count 4 decimal places to the right starting at the decimal point. The number next to the 8 is the first number cutoff, and it is greater than or equal to 5, so add 1 to the 8. This is stating that the number is nearer to 31.5639 than it is to 31.5638

```
    1234
31.5638|7
31.5639|
```

So the answer is 31.5639

Round 31.56387 to 2 d.p

Count 2 decimal places to the right starting at the decimal point. The number next to the 6 is the first number cutoff and it is smaller than 5, so no correction is needed. This is stating that the number is nearer to 31.56 than it is to 31.57

```
    12
31.56|387
31.56|
```

So the answer is 31.56

Rounding decimals to a given number of significant figures or decimal places

Round 0.0023824 to 4 sig. figs.

The first significant figure is the first non-zero number you come to working from left to right. In this case, 2. To keep 4 sig. figs. the first number cut off is 4 so there is no correction to make.

1 2 3 4

0.002382 | 4

0.002382 |

0.002382

Round 0.0023824 to 2 sig. figs.

The first significant figure is the first non-zero number you come to working from left to right. In this case, 2. To keep 2 sig. figs. the first number cut off is 8 so correct the 3 to a 4.

1 2

0.0023 | 824

0.0024 |

0.0024

Rounding larger numbers to a given number of significant figures

Round 365 892 to 2 sig. figs.

The first significant figure is the first non-zero number you come to working from left to right. In this case, 3. To keep 2 sig. figs. the first number cut off is 5, so correct the 6 to a 7 and put in zeros for all the numbers to the left of the decimal point.

1 2

365 892 |

370 000 |

370 000

Rounding measures

Write 36 548 cm in metres, correct to the nearest metre.

Change 36 548 cm to 365.48 m by dividing by 100.

Correcting to the nearest metre means cutting off all numbers after the decimal point. The first number cut off is a 4 so there is no need to make a correction.

36548 cm
365.48 m
365 m

365 m

Truncation of a decimal

Truncate 3.45699 after 3 decimal places.

Truncate means cut off without correction, so the two 9s are cut off but no correction is made to the 6.

3.456 ¦ 99
3.456 ¦

3.456

Inequalities related to rounding

A number, x, written correct to 1 d.p. is 3.6.

Which one of these inequalities gives the **complete** range of possible values of x ?

$3.55 \leq x \leq 3.649$

$3.55 \leq x \leq 3.65$

$3.55 < x \leq 3.65$

$3.55 \leq x < 3.65$

The correct range is $3.55 \leq x < 3.65$ as the smallest value x can take is 3.55 and it can take any value up to, but **not** including, 3.65 because 3.65 would be rounded up to 3.7.

Inequalities related to truncating

A number, x, written truncated to 1 d.p. is 3.6.

Which of these inequalities gives the complete range of possible values of x ?

$3.55 \leq x < 3.65$

$3.61 \leq x \leq 3.69$

$3.6 < x \leq 3.7$

$3.6 \leq x < 3.7$

Recall, a truncated number is one where numbers have been cut off without correction and the smallest possible number which could be truncated to 3.6 is 3.6 as $x = 3.6$ to 1 d.p.

The largest value which x could take would be just less than 3.7 so $x < 3.7$.

This means that the correct answer is $3.6 \leq x < 3.7$

M2.14

Use approximation to produce estimates of calculations, including expressions involving π or surds.

Estimating a calculation

Estimating a calculation is a useful check on the accuracy of a calculation, particularly to see if the magnitude of an answer is correct.

Numbers are approximated, usually to 1 or 2 significant figures to enable simple calculation; for example, 397 000 would be approximated to 400 000.

π is usually approximated to 3 or $\frac{22}{7}$

Surds are usually approximated to the nearest square number. For example, $\sqrt{15.6} \approx \sqrt{16}$ or 4.

Estimating a calculation

A calculator gives the value of $\sqrt{\frac{2\times26.37\times\pi}{0.00389}}$ as 2063.8 correct to 1 d.p. Is this likely to be correct?

We start by approximating the calculation in stages:

$2 \times 26.37 \times \pi \approx 2 \times 25 \times 3$

$2 \times 25 \times 3 = 150$

$0.00389 \approx 0.004$

Gives:

$$\sqrt{\frac{2\times26.37\times\pi}{0.00389}} \approx \sqrt{\frac{150}{0.004}} = \sqrt{\frac{150000}{4}} = \sqrt{37500}$$

But:

$$\sqrt{37500} \approx \sqrt{40000} = 200$$

So this suggests that the answer of 2063.8 has the decimal point in the wrong place and should be 206.38

M3. Ratio and proportion

M3.1

Understand and use scale factors, scale diagrams and maps.

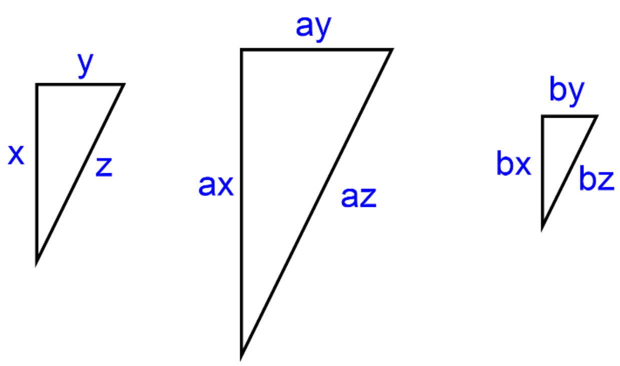
Similar shapes and scale factor

If two shapes are mathematically similar, then the lengths of the sides of one shape can be found from the lengths of the corresponding sides of the other shape by multiplying each length by the same number. This number is called the scale factor.

In the diagram the three triangles are similar. The first triangle has sides of length x, y and z, and a and b are constants and are the scale factors.

A scale factor can be less than 1.

In the diagram $a > 1$ and $b < 1$



Scale diagrams

A scale diagram is often smaller than the original drawing or object and is mathematically similar to it. In the diagrams above, each triangle is a scale drawing of the other two triangles.

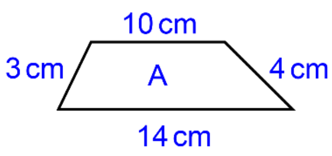
Maps

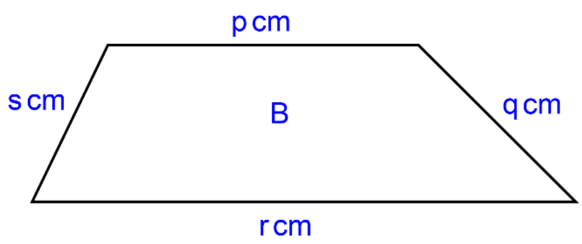
A map is a specific form of scale diagram. The scale of a map is often expressed as a ratio. If a map is drawn with 1 cm on the map representing 1 km in real life, then the scale of the map can be written either as:

1 cm represents 1 km or 1 : 100 000 (as 1km = 100 000 cm).

Similar shapes and scale factor

Trapezium B is an enlargement of Trapezium A with scale factor 3. Find the values of p, q, r and s.





If the scale factor is 3, then each length on Trapezium A is multiplied by 3 to find the equivalent length on Trapezium B.

$p = 3 \times 10 = 30$

$q = 3 \times 4 = 12$

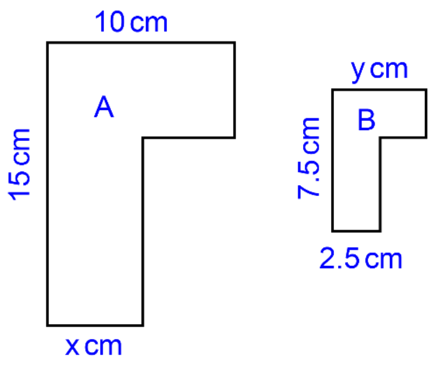
$r = 3 \times 14 = 42$

$s = 3 \times 3 = 9$

Scale diagrams

Figure B is a scale drawing of Figure A.

Find the values of x and y.



7.5cm is half of 15cm so the scale factor is $\frac{1}{2}$

Lengths on Figure A are $\frac{1}{2}$ the lengths on Figure B so the scale factor is $\frac{1}{2}$

$y = 10 \times \frac{1}{2} = 5$

$\frac{1}{2}x = 2.5$ so $x = 2.5 \times 2 = 5$

Maps

A map of a village is drawn with a scale of 1 : 25 000

On the map the distance from the bank to the post office is 3cm. How far is this in metres?

3cm on the map is 3 × 25 000 cm in the village. 3 × 25 000 = 75 000 cm

75 000 cm = 750 m (There are 100 cm in a metre.)

In the village the distance from the supermarket to the bank is 200 metres. How many mm is this on the map?

200 m = 20 000 cm

20 000 cm in the village is 20 000 ÷ 25 000 cm on the map

20 000 ÷ 25 000 = 0.8 cm

0.8 cm = 8 mm (There are 10 mm in 1 cm)

M3.2

Express a quantity as a fraction of another, where the fraction is less than 1 or greater than 1.

One quantity can be expressed as a fraction of another

$\frac{x}{y}$ is x expressed as a fraction of y

To express one quantity as a fraction of another, both quantities must be in the same units. This will normally be the smaller of the two units.

Expressing one quantity as a fraction of another – fraction less than 1

Express 200 g as a fraction of 1 kg.

The quantities are in different units so change 1 kg to 1000 g.

200 g as a fraction of 1000 g is $\frac{200}{1000} = \frac{1}{5}$

Expressing one quantity as a fraction of another – fraction greater than 1

Express 1 litre as a fraction of 450 ml.

Start by writing in the same units.

1 litre = 1000 ml

1000 ml as a fraction of 450 ml is $\frac{1000}{450} = \frac{20}{9} = 2\frac{2}{9}$

M3.3

Understand and use ratio notation.

Ratio notation

If a bag contains x red sweets and y yellow sweets, then the ratio of red sweets to yellow sweets is $x : y$ which can also be written as x to y.

Simplifying ratios

Both sides of the ratio can be multiplied or divided by the same positive number without changing the ratio.

Comparing using ratios

To compare quantities as ratios, the units must be the same.

Ratio notation

Jana has 63 cm of red ribbon and 59 cm of blue ribbon. What is the ratio of the length of red ribbon to the total length of blue and red ribbon?

The total length of ribbon is 63 + 59 = 122 cm

The ratio of the length of the red ribbon to the total length of red and blue ribbon is 63 : 122

Simplifying ratios

A mixture contains 2.5 kg of flour and 1.5 kg of sugar.

What is the ratio of flour to sugar, by weight, in the mixture? Give your answer as a ratio of integers.

The ratio is 2.5 : 1.5

To write this as an integer ratio, multiply both sides of the ratio by 2 to get:

2.5 : 1.5 is the same as 2.5 × 2 : 1.5 × 2 which is 5 : 3

Comparing using ratios

A mixture contains 2.5 kg of flour and 750 g of sugar.

What is the ratio of flour to sugar, by weight, in the mixture? Give your answer as a ratio in its lowest integer terms.

2.5 kg = 2500 g

The ratio of weights is 2500 : 750 which is the same as 2500 ÷ 250 : 750 ÷ 250 or 10 : 3

M3.4

Divide a given quantity into two (or more) parts in a given *part* **:** *part* ratio.

Express the division of a quantity into two parts as a ratio.

Dividing in a given ratio

To divide a quantity Q in the ratio $x : y$, first divide Q by $x + y$ to find the value of one part.

Multiply the value of one part by x to find the value of x parts and then by y to find the value of y parts.

Check that the values of the x parts and the y parts add up to Q.

Expressing a division into parts as a ratio

To express the division of a quantity into two parts as a ratio, first make sure that both parts are in the same units and then use ratio notation to relate them.

Dividing in a given ratio

Divide £450 in the ratio 11 : 7

11 + 7 = 18 parts

1 part is £450 ÷ 18 = £25

11 parts are £25 × 11 = £275

7 parts are £25 × 7 = £175

Check: £275 + £175 = £450

Expressing a division into parts as a ratio

A piece of ribbon is divided into 2 pieces, A and B.

A is 125 cm long and B is 2.75 m long. What is the ratio of the lengths of A and B?

First put the lengths of A and B into the same units. A is 125 cm and B is 275 cm

The ratio of the lengths of A : B is 125 : 275.

Dividing both sides of the ratio by 25, we can simplify this to 5 : 11

M3.5

Apply ratio to real contexts and problems, such as those involving conversion, comparison, scaling, mixing and concentrations.

Express a multiplicative relationship between two quantities as a ratio or a fraction.

Ratio can be applied to problems involving:

- conversion
- comparison
- scaling
- mixing
- concentrations

A multiplicative relationship between two quantities can be expressed as a ratio or fraction

First, create an equation.

If there is 'a' times as much of X in a mixture as there is of Y, then $aY = X$ and the ratio (quantity of X) : (quantity of Y) is $a : 1$

Writing this as a fraction gives

$$\frac{quantity\ of\ X}{quantity\ of\ Y} = \frac{a}{1} = a$$

Application of ratio to conversion

The conversion ratio of dinars to dollars is 11 : 2. How many dollars can you get for 2530 dinars?

Every 11 dinars are worth 2 dollars.

2530 ÷ 11 = 230

230 lots of 11 dinars are worth 230 lots of 2 dollars = 460 dollars

Application of ratio to comparison

In a mixture of orange purée, mango purée and water, the ratio, by volume, of orange purée to water is 3 : 5 and the ratio of mango purée to water is 11 : 15.

What is the ratio in this mixture, by volume, of orange purée to mango purée?

To find the ratio of two quantities (in this case orange and mango purée) which are both given as a ratio of the same third quantity (in this case water), you need the linking ratio to be the same.

Orange to water is 3 : 5 and water to mango is 15 : 11. To eliminate the water ratio, you need the number of parts of water to be the same in both ratios, so write 3 : 5 as 3 × 3 : 5 × 3 or 9 : 15

Orange to water is 9 : 15 and water to mango is 15 : 11, so orange : mango is 9 : 11

Application of ratio to scaling

A scale drawing is made of a park using a ratio of 1 : 10 000

The length of the park on the scale drawing is 30 cm. What is the length of the park in km?

The ratio of 1 : 10 000 means that every measurement on the drawing is multiplied by 10 000 to get the real measurement.

30 cm on the map is 30 × 10 000 cm in the real park.

30 × 10 000 cm = 300 000 cm = 3000 m = 3 km

Application of ratio to mixing

A mixture, X, contains orange concentrate and water in the ratio 3 : 7

A second mixture, Y, contains orange concentrate and water in the ratio 1 : 4

1 l of X and 2 l of Y are mixed together to form a mixture, Z.

What is the ratio of orange concentrate to water in Z?

Method 1

Divide the volume of X into 3 + 7 = 10 parts. Each part is 1000 ÷ 10 = 100 ml so there are 3 × 100 = 300 ml of orange and 7 × 100 = 700 ml of water.

Divide the volume of Y into 1 + 4 = 5 parts. Each part is 2000 ÷ 5 = 400 ml so there are 1 × 400 = 400 ml of orange and 4 × 400 = 1600 ml of water.

In the mixture there are 300 ml + 400 ml = 700 ml of orange and 700 ml + 1600 ml = 2300 ml of water. The ratio of orange to water in the mixture is 700 : 2300 or 7 : 23

Method 2

3 : 7 gives 10 parts.

1 : 4 can be written as 2 : 8, also giving 10 parts.

To use this method, both mixtures must be written as ratios with the same number of parts. The ratio in the new mixture is (3 + (2 × 2)) : (7 + (2 × 8)) or 7 : 23

Application of ratio to concentrations

When a gas is compressed to become a liquid, the ratio of the original volume of the gas to the final volume of the liquid is 100 : 3

How many litres of gas must be compressed to form 1 litre of the liquid?

The ratio means that 100 litres of gas are used to produce 3 litres of liquid.

Therefore, the quantity of gas needed to produce 1 litre of liquid is $100 \div 3 = 33\frac{1}{3}$ litres

Expressing a multiplicative relationship between two quantities as a ratio or fraction

In a fruit smoothie containing only apple juice and mango juice there is twice as much apple juice as mango juice.

a) What is the ratio of apple juice to mango juice in the mixture?

b) What fraction of the smoothie is apple juice?

Let a be the quantity of apple juice in the smoothie and m be the quantity of mango juice.

a) If there is twice as much apple juice then:

a = 2m (i)

To find the ratio of apple to mango:

a : m Substitute from (i)

= 2m : m

= 2m ÷ m : m ÷ m Divide through by m

= 2 : 1 Simplify

b) If the ratio of apple to mango is 2 : 1, then the fraction of apple in the smoothie is $\frac{2}{2+1} = \frac{2}{3}$

M3.6

Understand and use proportion.

Relate ratios to fractions and to linear functions.

Simple proportion

If x bars of chocolate cost £y then 1 bar of the same chocolate costs £$\frac{y}{x}$ and a identical bars of the same chocolate costs £$\frac{ay}{x}$

Relating ratios to fractions

If the ratio of x to y in a mixture consisting only of x and y is $a : b$, then the fraction of x in the mixture is $\frac{a}{a+b}$

Relating ratios to linear functions

If the ratio of x to y in a mixture consisting only of x and y is $a : b$, then the relationship between x and y is $bx = ay$ or $y = \frac{bx}{a}$

Simple proportion

If 12 boxes of Chocky Biscuits cost £42, how much do 20 boxes of Chocky Biscuits cost?

Method 1

Unitary method

12 boxes cost £42

First, find the cost of one box.

1 box costs £42 ÷ 12 (unless the quotient is an integer, do not divide out at this stage) Now, multiply to find the cost of 20 boxes.

20 boxes cost £42 ÷ 12 × 20

$42 \div 12 \times 20 = \frac{42\times20}{12}$

$= \frac{7\times20}{2}$ [Divide numerator and denominator by 6]

$= \frac{7\times10}{1}$ [Divide numerator and denominator by 2]

= 70

So 20 boxes cost £70.

Method 2

Using the HCF

The HCF of 12 and 20 is 4, so find the cost of 4 boxes.

12 boxes cost £42, so 4 boxes cost £42 ÷ 3 = £14

20 boxes cost £14 × 5 = £70

Relating ratios to fractions

A packet of sweets contains only yellow, green and red sweets. If the ratio of yellow sweets to green sweets in a mixed pack of sweets is 3 : 4 and the ratio of green sweets to red sweets is 6 : 9, what fraction of the sweets in the pack is yellow?

To find the fraction you need the ratio yellow : green : red. To combine the two ratios, either the ratio 3 : 4 must be expressed as x : 6 or the ratio 6 : 9 must be expressed as 4 : y to give a common link between the two ratios.

3 : 4 is the same as $3 \times \frac{3}{2} : 4 \times \frac{3}{2}$ which is 4.5 : 6.

The ratio of yellow : green : red is 4.5 : 6 : 9 or 9 : 12 : 18 (multiplying through by 2)

The fraction of the sweets that are yellow is $\frac{9}{9+12+18} = \frac{9}{39} = \frac{3}{13}$

Relating ratios to linear functions

The ratio $x : y$ is 2 : 3

Write y as a linear function of x.

First convert the ratios into fractions:

$\frac{x}{y} = \frac{2}{3}$

$\frac{x}{y} \times y = \frac{2}{3} \times y$ [Multiply both sides of the equation by y]

$x = \frac{2}{3}y$

$x \times 3 = \frac{2}{3}y \times 3$ [Multiply both sides of the equation by 3]

$3x = 2y$ [Simplify]

$3x \div 2 = 2y \div 2$ [Divide both sides of the equation by 2]

$y = \frac{3x}{2}$ [Simplify]

M3.7

Identify and work with fractions in ratio problems.

Ratios and fractions

If the ratio of $x : y$ is $p : q$ then $\frac{x}{y} = \frac{p}{q}$

Example

A bag contains y yellow counters, g green counters and r red counters.
The ratio of $r : y$ is 2 : 3
The ratio of $g : r$ is 4 : 5
What is the ratio $g : y$?

Method 1

$\frac{r}{y} = \frac{2}{3}$ (i)

$\frac{g}{r} = \frac{4}{5}$ (ii)

To find $g : y$, first find $\frac{g}{y}$ using $\frac{g}{y} = \frac{g}{r} \times \frac{r}{y}$

This gives: $\frac{g}{y} = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$

Therefore $g : y$ is 8 : 15

Method 2

To combine the two ratios, either the ratio 2 : 3 must be expressed as x : 4 or the ratio 4 : 5 must be expressed as 3 : y to give a common link between the two ratios.

2 : 3 is the same as $2 \times \frac{4}{3} : 3 \times \frac{4}{3}$ which is $\frac{8}{3}$: 4

The ratio of yellow : green : red is $\frac{8}{3}$: 4 : 5 or 8 : 12 : 15 (multiplying through by 3)

Therefore $g : y$ is 8 : 15

M3.8

Define percentage as 'number of parts per hundred'.

Interpret percentages and percentage changes as a fraction or a decimal, and interpret these multiplicatively.

Express one quantity as a percentage of another.

Compare two quantities using percentages.

Work with percentages greater than 100%.

Solve problems involving percentage change, including percentage increase/decrease, original value problems and simple interest calculations.

Percentage as parts per hundred

Percentage means 'number of parts per hundred'.

If 12 sweets out of every 50 in a mixture of sweets are red, then this is the same as 24 sweets per hundred, so 24 per cent of the sweets are red. 24 per cent is written as 24%.

Percentages as fractions or decimals used multiplicatively

24% means 24 per hundred. As a fraction this is $\frac{24}{100}$ (or $\frac{6}{25}$) and as a decimal this is 0.24

Calculating 24% of 50 means multiply 24% by 50. To do this, it is necessary to change 24% into a fraction or decimal,

giving $\frac{24}{100} \times 50 = 12$ or $0.24 \times 50 = 12$

Expressing one quantity as a percentage of another

To express 18 as a percentage of 30, first write 18 as a fraction of 30, which is $\frac{18}{30}$ or $\frac{3}{5}$

Then change $\frac{3}{5}$ into a percentage by finding $\frac{3}{5}$ of 100

$\frac{3}{5}$ × 100 = 60 so 18 is 60% of 30.

Comparing two quantities using percentages

Bag A contains 200 sweets, of which 54 are red. Bag B contains 150 sweets, of which 42 are red. Bag B has the higher percentage of red sweets because:

$\frac{54}{200} = \frac{27}{100} = 27\%$

and $\frac{42}{150} = \frac{84}{300} = \frac{28}{100} = 28\%$

Percentages greater than 100%

If a salary increases by 10% then it is now 110% of the original salary.

Percentage change = $\frac{\text{actual change}}{\text{original amount}}$ × 100 so:

percentage increase (or decrease) = $\frac{\text{actual increase [or decrease]}}{\text{original amount}}$ × 100

percentage profit (or loss) = $\frac{\text{actual profit [or loss]}}{\text{original price}}$ × 100

Finding the original price after an increase or decrease

Treat the original price as 100%.

If a price is decreased by x%, if the original price is 100%, the new price is (100−x)%.

If the new price is Q, then the original price is $\frac{Q}{100-x}$ × 100

If a price is increased by y%, if the original price is 100%, the new price is (100 + y)%.

If the new price is R, then the original price is $\frac{R}{100+y}$ × 100

Simple interest

If the interest on a savings account in the bank is r % per annum (per year) then the simple interest gained in 5 years is 5 × r % = 5r %

Percentage as parts per hundred

If 3 out of every 8 pupils in a school eats more than their 5 a day of fruit and vegetables, what percentage is that?

3 out of 8 is 300 out of 800, which is 300 ÷ 8 out of 100

300 ÷ 8 = 37.5, so 3 out of 8 is 37.5 out of 100 or 37.5%

Percentages as fractions or decimals used multiplicatively

What is 35% of 220?

35% $= \frac{35}{100} = \frac{7}{20}$

35% of 220 is $\frac{7}{20}$ × 220 = 7 × 11 = 77

Expressing one quantity as a percentage of another

What percentage of 80 is 24?

24 as a fraction of 80 is $\frac{24}{80} = \frac{3}{10}$

$\frac{3}{10} \times 100 = 30$, so 24 is 30% of 80.

Comparing two quantities using percentages

If Tank A has 300 ml of oxygen in 2.5 l of gas and Tank B has 840 ml of oxygen in 7.5 l of gas, which gas has the higher concentration of oxygen?

2.5 l = 2500 ml and 7.5 l = 7500 ml.

The percentage of oxygen in the gas in Tank A is $\frac{300}{2500} \times 100\% = 12\%$

The percentage of oxygen in the gas in Tank B is $\frac{840}{7500} \times 100\% = \frac{840}{75}\% = \frac{56}{5}\% = 11.2\%$

The concentration of oxygen is greater in the gas in Tank A.

Percentages greater than 100%

A carpenter makes a chair for a total cost of £22. She sells the chair for £77. What percentage is this of her cost price?

77 as a percentage of 22 is $\frac{77}{22} \times 100\% = \frac{7}{2} \times 100\% = 350\%$

77 is 350% of 22

Percentage change

A carpenter makes a chair for a total cost of £22. She sells the chair for £77.

What is her percentage profit?

Her actual profit is £77 − £22 = £55

Her percentage profit is $\frac{\text{actual profit}}{\text{original price}} \times 100\% = \frac{55}{22} \times 100\% = 250\%$

Finding the original price after an increase or decrease

After a percentage increase of 20% the price of a pair of shoes was £96.

What is the original price of the pair of shoes?

Let the original price be 100%

If the original price was 100%, then the new price is 100% + 20% = 120%

If 120% is £96, then $\frac{£96}{120}$ is 1%

So 100% is $\frac{£96}{120} \times 100 = £80$

Simple interest

John puts £400 into a savings account.

The simple interest on his savings is 3.5% per annum (per year).

If he leaves his money in his savings account for 8 years, how much interest will he receive?

Method 1

Each year he will receive in £ interest of $\frac{3.5}{100}$ × £400 = £14

In 8 years he will receive 8 × £14 = £112

Method 2

3.5% interest in one year is 8 × 3.5% in 8 years which is 28%.

28% of £400 is $\frac{28}{100}$ × £400 = £112

M3.9

Understand and use direct and inverse proportion, including algebraic representations.

Recognise and interpret graphs that illustrate direct and inverse proportion.

Set up, use and interpret equations to solve problems involving direct and inverse proportion (including questions involving integer and fractional powers).

Understand that x is inversely proportional to y is equivalent to x is proportional to $\frac{1}{y}$

Note: The sign for 'is proportional to' is $\propto$

Direct proportion

If one chocolate bar costs £6 then 6 bars cost £36 and x bars cost £6x. There is no reduction in price per bar for bulk buying. The number of chocolate bars and the total price are in direct proportion, one increases in proportion to the increase in the other. If y is the cost of x bars then the direct proportion is written as $y \propto x$ or as the equation $y = 6x$ which is a straight line graph through the origin.

In general, if y is directly proportional to x, we can write the relationship as $y \propto x$ or $y = kx$ where k is a constant.

You could use any letter, apart from the letters used for the variables, to represent the constant.

Graphs illustrating direct proportion

If a graph is plotted of y against x and they are in direct proportion, then the graph will be a straight line through the origin.

Inverse proportion

If two variables are in inverse proportion to each other, then one variable increases as the other decreases.

For example, if the thickness, h, of ice on a pond is inversely proportional to the air temperature, t, then the ice gets thicker as the temperature gets lower. h is inversely proportional to t is written as $h \propto \frac{1}{t}$ and so $h = \frac{k}{t}$ where k is a constant.

Graphs showing inverse proportion

If a graph is plotted of y [vertical axis] against x [horizontal axis] and they are inversely proportional to each other, then the graph is of the form shown in the diagram.

Proportion involving integer and fractional indices

If $y \propto x^n$ then $y = kx^n$ for integer and fractional values of n.

Direct proportion

On days when the outside temperature is above 20°C, the number of ice creams sold in a shop is proportional to (t − 15) where t°C is the outside temperature.

When the temperature is 20°C the shop sells 30 ice creams.

How many ice creams will the shop sell on a day when the temperature is 28°C?

Let the number of ice creams sold be n.

$n \propto (t - 15)$ so

$n = k(t - 15)$ (i)

First find k: Substitute $t = 20$, $n = 30$ into (i)

$30 = k(20 - 15)$

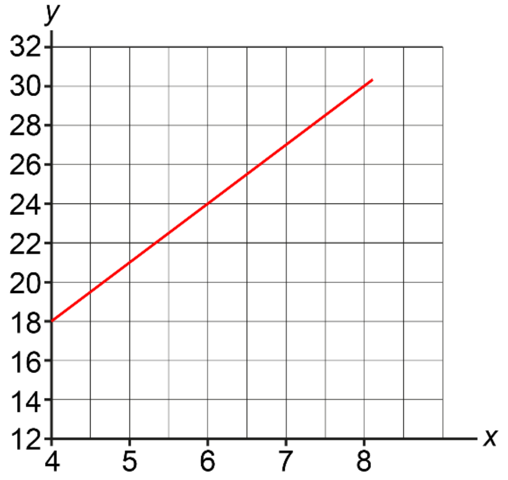
$30 = 5k$ so $k = 6$

Substitute $t = 28$ and $k = 6$ into (i) $n = 6(28 - 15) = 78$

Graphs illustrating direct proportion

The diagram shows part of a graph of values of x plotted against values of y.

Is y directly proportional to x?



Method 1

If a graph is plotted of y against x and they are in direct proportion then the graph will be a straight line through the origin. Here there is a straight line graph, but it is not through the origin.

Method 2

If $y \propto x$ then: $y = kx$ (i)

From the graph when $x = 4$, $y = 18$

Substitute into (i) to find k.

$18 = 4k$ therefore $k = 4.5$

Check another point on the graph for example $x = 6$, $y = 24$

If $y \propto x$ then $24 = 6k$ but $24 \neq 6 \times 4.5$ so y is not directly proportional to x.

Inverse proportion

The number, n, of pairs of gloves sold in a store in a day is inversely proportional to the outside temperature t°C for

$1 \leq t \leq 10$

When t = 5, n = 15

How many pairs of gloves are sold on a day when t = 3?

If n is inversely proportional to t then $n \propto 1/t$ or:

$n = \frac{k}{t}$ (i)

First find k:

Substitute t = 5, n = 15 into (i)

$15 = \frac{k}{5}$ therefore k = 15 × 5 = 75

When t = 3, $= \frac{75}{3} = 25$

Graphs showing inverse proportion

The diagram shows part of a graph of values of x plotted against values of y.

Is it reasonable to say that y could be inversely proportional to x?

If y is inversely proportional to x then $y \propto \frac{1}{x}$ or:

$y = \frac{k}{x}$ (i)

Find k from different points on the line. If the value of k is the same for all points then y is inversely proportional to x.

When x = 1, y = 12 and k = 12

When x = 2, y = 6 and k = 12

When x = 3, y = 4 and k = 12

When x = 12, y = 1 and k = 12

When x = 6, y = 2 and k = 12 When x = 4, y = 3 and k = 12

Hence, it is reasonable to assume that y is inversely proportional to x but without checking all points it is not possible to be certain.

Proportion involving integer and fractional indices

The number, n, of bottles of water sold at a shopping centre in a day is proportional to the square of the average day temperature t (°C) for $10 \leq t \leq 30$

On a day when the average temperature is 12°C, 72 bottles of water are sold.

How many bottles of water are sold on a day when the average temperature is 24°C?

If n is proportional to the square of t then $n \propto t^2$ or:

$n = kt^2$ (i)

First find k:

Substitute $n = 72$, $t = 12$ into (i)

$72 = k \times 144$ so $k = 0.5$

When $t = 24$, then $n = 0.5 \times 24^2 = 288$

M3.10

Compare lengths, areas and volumes using ratio notation.

Understand and make links to similarity (including trigonometric ratios) and scale factors.

Definition of similarity

If two shapes are mathematically similar then one shape is an enlargement of the other – they are the same shape, have the same angles in the same order and corresponding sides are in the same ratio.

Area ratio and volume ratio from a linear scale factor

If two shapes A and B are mathematically similar and the lengths of the sides of B are x times the lengths of the corresponding sides of A, then:

- the area of the surfaces of B are x^2 times the areas of the corresponding surfaces of A
- the volume of B is x^3 times the volume of B.

Area ratio and volume ratio from a linear ratio

If two shapes A and B are mathematically similar and the ratio of corresponding lengths on shape A to those on shape B is $x : y$ then:

- the ratio of corresponding areas is $x^2 : y^2$ the ratio of the volumes is $x^3 : y^3$

Linear ratio from the area ratio

If two shapes A and B are mathematically similar and the area of B is x times the area of A, then the lengths of shape B are $\sqrt{x}$ times the corresponding lengths of shape A.

If two shapes A and B are mathematically similar and the ratio of the area of B to the area of A is $x : y$ then the ratio of the lengths of shape B to the corresponding lengths of shape A is $\sqrt{x} : \sqrt{y}$

Linear ratio and area ratio from the volume ratio

If two shapes A and B are mathematically similar and the volume of B is x times the volume of A, then:

- the lengths of shape B are $\sqrt[3]{x}$ times the corresponding lengths of shape A
- the areas on shape A are $\left(\sqrt[3]{x}\right)^2$ times the corresponding areas on shape B.

If two shapes A and B are mathematically similar and the ratio of the volume of B to the volume of A is $x : y$ then:

- the ratio of the lengths of shape B to the corresponding lengths of shape A is $\sqrt[3]{x} : \sqrt[3]{y}$
- the ratio of the areas of shape B to the corresponding areas of shape A is $\left(\sqrt[3]{x}\right)^2 : \left(\sqrt[3]{y}\right)^2$

Area ratio and volume ratio from a linear scale factor

X is a cube of side x cm.

Y is a cube of side $3x$ cm.

If the volume of X is 100 cm^3 then what is the volume of Y?

Each length of X is multiplied by 3 ($3x \div x = 3$)

So the volume of Y is 3^3 times the volume of X.

So the volume of Y is $3^3 \times 100 = 2700$ cm^3.

Area ratio and volume ratio from a linear ratio

The ratio of the diameters of two spheres A and B is 2 : 5

The surface area of A is 250 cm^2. Find the surface area of B.

Diameter is a length.

The ratio of the lengths is 2 : 5 so the ratio of the areas is $2^2 : 5^2$ which is 4 : 25 or 1 : $\frac{25}{4}$

The area of B is $\frac{25}{4} \times 250$ cm^2 = $\frac{6250}{4}$ = 1562.5 cm^2

Linear ratio from the area ratio

A and B are mathematically similar shapes.

The ratio of the area of shape A to the area of shape B is 9 : 25

The length of shape A is 21 cm, what is the length of shape B?

If the ratio of the areas is 9 : 25 then the ratio of the lengths is $\sqrt{9} : \sqrt{25}$ or 3 : 5

If the length of A is 21 cm then the length of B is 21 ÷ 3 × 5 = 35 cm.

Linear ratio and area ratio from the volume ratio

The volume of a cylinder, P, is 64 times the volume of a mathematically similar cylinder, Q.

The surface area of Q is 300 cm^2.

What is the surface area of P?

To go from a volume ratio to an area ratio you must find the length ratio first.

The ratio of the volumes is 1 : 64 so the ratio of corresponding lengths is $\sqrt[3]{1}$: $\sqrt[3]{64}$ which is 1 : 4

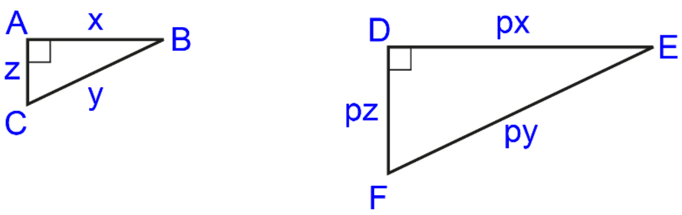
If the length ratio is 1 : 4 then the area ratio is $1^2 : 4^2$ or 1 : 16

If the area of Q is 300 cm^2 then the area of P is 300 × 16 = 4800 cm^2.

Similarity, trigonometric ratio and scale factor

If ABC is a right-angled triangle with sides of length x, y and z and DEF is an enlargement of triangle ABC with scale factor p, then the angles and trigonometrical ratios are preserved – they do not depend on the scale factor.

In the diagram:



$\tan C = \frac{x}{z}$ and $\tan F = \frac{px}{pz} = \frac{x}{z} = \tan C$

So angle C = angle F as they are both acute angles.

The other trigonometrical ratios and angles work in the same way.

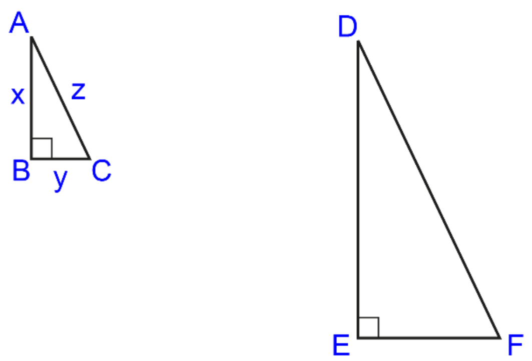
Similarity, trigonometric ratio and scale factor

Triangle DEF is an enlargement of triangle ABC with scale factor 4.

The sides of triangle ABC have lengths x, y and z, with y < x < z

In triangle DEF, EF < ED < DF

What is the sine of angle DFE in terms of x, y and z?



As triangle DEF is an enlargement of triangle ABC the two triangles are similar so the angles of the two triangles are equal.

Angle A = angle D as they are opposite the smallest side of the triangles,

so angle C = angle F

so sin F = sin C = $\frac{x}{z}$

M3.11

Set up, solve and interpret the answers in growth and decay problems, including compound interest, and work with general iterative processes.

Exponential growth and decay

Problems of growth and decay will involve a rate of growth or decay where the quantity is multiplied by the same number in each time period.

If the size of the population initially (at time = 0) is q and the size of the population is multiplied by a factor x every hour, then after n hours the size of the population is qx^n. For growth $x > 1$ and for decay $0 < x < 1$. If $x = 1$ the population is static.

Examples are bacteria colonies which treble in size every hour or a substance losing half of its radioactivity every 6 hours.

Compound interest

If an initial sum, P, is in an account and the rate of compound interest is r% per annum, then after n years the amount, A, of money in the account is $A = P\left(1 + \frac{r}{100}\right)^n$

Per annum means per year.

Iterative processes

An iterative process is one where a basic set of instructions or rules are used again and again, usually over time. Compound interest is an example of an iterative process as interest is added in the same way again and again over time.

Growth and decay

In an epidemic the number of patients in a town trebles every day. At the end of day 1 there are 50 patients.

a) How many patients are there at the end of day 4?

When the number of patients reaches 4000, a cure is found and given to the patients. The number of patients decreases exponentially. After 4 days of using the cure, there are 500 patients.

b) How many patients will there be after 6 days of using the cure?

a) At the end of day 4 there will be 50×3^3 patients.

$50 \times 3^3 = 50 \times 27 = 1350$

b) If the rate of decay is r then after 4 days there will be $4000r^3$ patients

$4000r^3 = 500$ so $\quad r^3 = \frac{500}{4000} = \frac{1}{8}\ so\ r = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

After 6 days there will be $4000r^5$ patients.

$4000r^5 = 4000 \times \left(\frac{1}{2}\right)^5 = 4000 \times \frac{1}{32} = 125$

Compound interest

£1000 is invested in a savings account at 4% per annum compound interest for 5 years.

How much interest has been received after 5 years? Give your answer to the nearest penny.

In 5 years the amount of money in the account is £1000$\left(1+\frac{4}{100}\right)^5$ = £1000 × 1.04^5 = £1216.65 to the nearest penny.

The interest earned is £1216.65 − £1000 = £216.65

M4. Algebra

M4.1

Understand, use and interpret algebraic notation; for instance: ab in place of $a \times b$; $3y$ in place of $y + y + y$ and $3 \times y$; a^2 in place of $a \times a$; a^3 in place of $a \times a \times a$; $a^2 b$ in place of $a \times a \times b$; a/b in place of $a \div b$.

Using letters and numbers in algebra

Numbers, letters, and brackets are multiplied or divided together, to make algebraic terms

Multiplying

$a \times b$ can be written as ab without the '×' and any spaces
$p \times q \times r = pqr$
$a \times a$ can be written as a^2
$p \times p \times p = p^3$
$p \times p \times q = p^2q$
$4 \times b = 4b$
$a + a + a = 3 \times a = 3a$
$5 \times p \times p \times p = 5p^3$

When writing terms with numbers and letters, the number goes first, then the letters in alphabetical order, for example, $6pqr$

Dividing

$a \div b$ can be written as $\frac{a}{b}$

Brackets

$(ab)^2$ means $ab \times ab = a^2 \times b^2$

M4.2

Use index laws in algebra for multiplication and division of integer, fractional, and negative powers.

Index notation

a^5 means a raised to the power 5, where a is the base, and 5 is the power or index (plural indices).
$a \times a \times a \times a \times a = a^5$

Multiplication

To multiply powers of the same base, add the indices:

$a^m \times a^n = a^{m+n}$

Any coefficients (5 is the coefficient of p in $5p$) are multiplied, for example:

$5p^2q \times 2pq^2 = 10p^3q^3$

You should be able to: Multiply powers of the same base and simplify expressions.

Division

To divide powers of the same base, subtract the indices:

$a^m \div a^n = a^{m-n}$

Numerical coefficients are divided first, for example:

$20p^3q^2 \div 2p^2q = 20/2p^{3-2}q^{2-1} = 10pq$

You should be able to: Divide powers of the same base and simplify expressions.

Raising terms to a further power

To raise a term to a further power, multiply the powers.

$(a^m)^n = a^{m \times n} = (a^n)^m$

The powers of all numbers and letters in the term must be multiplied, for example:
$(4ab^2)^3 = 4^3a^3b^{2\times3} = 64a^3b^6$

You should be able to: Raise a term to a further power.

Also:

- Any base raised to the power ZERO is equal to 1. In general, $a^0 = 1$, for all non-zero values of a.
- Any base to the power of 1 is just the letter or number itself so $a^1 = a$

Fractions

The power applies to both the numerator and the denominator:

In general, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

You should be able to:

- Raise fractions to a power and simplify expressions.

Negative powers

A base raised to a negative power can be written as a fraction with a numerator of 1 over a denominator of the base to the positive power:

$a^{-m} = \frac{1}{a^m}$

You should be able to:

- Raise a base to a negative power.
- Raise a base to a negative power when the coefficient is not 1.

Fractional powers

The power $1/2$ is the same as the square root, so $a^{\frac{1}{2}} = \sqrt{a}$

The power $1/3$ is the same as the cube root, so $a^{\frac{1}{3}} = \sqrt[3]{a}$

The power $1/4$ is the same as the fourth root etc.

In general: $a^{\frac{1}{b}} = \sqrt[b]{a}$

And also:

$$a^{-\frac{1}{b}} = \frac{1}{\sqrt[b]{a}}$$

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$$

You should be able to:

- Simplify expressions with a fractional power.
- Simplify expressions with a fractional power, evaluating any numbers as needed.

Multiply powers of the same base and simplify expressions

Simplify $4a^2 \times 2a^3$

Remember to multiply the numbers and add the indices.

$4a^2 \times 2a^3 = 4 \times 2 \times a^{3+2} = 8a^5$

Divide powers of the same base and simplify expressions

Simplify $12a^6 \div 2a^3$

Remember to divide the numerical coefficients and subtract the indices (powers).

$12a^6 \div 2a^3 = 6a^{6-3} = 6a^3$

Raise a power to a further power and simplify expressions

Simplify

$(2p^3)^5$

Remember to multiply the powers of the numerical coefficient and the base.

$(2p^3)^5 = 2^5p^{3\times 5} = 32p^{15}$

Raise fractions to a power

Simplify $\left(\frac{2p^2}{3q^3}\right)^4$

The numerator and the denominator are both raised to the power outside the brackets. Remember to multiply the powers of the numbers and the base.

$$\left(\frac{2p^2}{3q^3}\right)^4 = \frac{(2p^2)^4}{(3q^3)^4} = \frac{2^4 p^{2\times 4}}{3^4 q^{3\times 4}} = \frac{2^4 p^8}{3^4 q^{12}} = \frac{16p^8}{81q^{12}}$$

Raise a base to a negative power and simplify expressions

Simplify $(4q^3)^{-4}$

The number and base are both raised to the negative power and can be written as 1 over the positive power of these.

$$(4q^3)^{-4} = \frac{1}{4^4 q^{3\times 4}} = \frac{1}{256q^{12}} \quad \left(\text{or } \frac{1}{256}q^{-12}\right)$$

Simplify expressions with a fractional power

Simplify $(25y^4)^{\frac{1}{2}}$

Remember the power $1/2$ means the square root of the number or base.

$$(25y^4)^{\frac{1}{2}} = 25^{\frac{1}{2}}\ (y^4)^{\frac{1}{2}} = 5\ y^{4\times\frac{1}{2}} = 5y^2$$

Simplify expressions involving negative and fractional powers

Simplify $(8a^{-3})^{\frac{1}{3}}$

Remember the power $\frac{1}{3}$ means the cube root of the number or base.

$$(8a^{-3})^{\frac{1}{3}} = (8)^{\frac{1}{3}}\,(a^{-3})^{\frac{1}{3}} = 2a^{-1} = \frac{2}{a}$$

M4.3

Substitute numerical values into formulae and expressions, including scientific formulae.

Understand and use the concepts and vocabulary: *expressions*, *equations*, *formulae*, *identities*, *inequalities*, *terms* and *factors*.

Definitions

If a rectangle has length l, width w and perimeter P:

P = 2l + 2w is a formula for the perimeter of the rectangle. It relates the variables P, l and w.

2l + 2w is an expression for the perimeter of the rectangle.

This expression has two terms which are 2l and 2w.

An expression might only have one term.

An expression does not have an = sign.

An equation always has an = sign and is true only for certain values of a variable.

An identity is always true. An identity can be shown by using the ≡ sign.

$3x + 1 = 7$ is an equation which is true for $x = 2$.

$6 \equiv 2 + 4$ and $3x + 2x - 4x \equiv x$ are identities as they are always true.

An inequality describes the relative size of two expressions:
less than (<), less than or equal to (≤), greater than (>), greater than or equal to (≥), not equal to (≠).

$3 + 4 < 8$, $2x + 6 \leq 3x$, $x^2 > 4$, $4x - 3 \geq -2$ and $6 + 3 \neq 7$ are all inequalities.

A factor of a quantity or expression divides exactly into that quantity or expression without leaving a remainder.

6 is a factor of 36, xy is a factor of $3x^2y^3$,

$(x + 1)$ is a factor of $x(x + 1)(x + 3)$

Substitution

Numerical values can be substituted into formulae and expressions to evaluate the expression or formula. The rules of BIDMAS apply.

Factors of algebraic expressions

List the factors of p^3q^2

$1, p, q, p^2, pq, q^2, p^3, p^2q, pq^2, p^3q, p^2q^2, p^3q^2$

The factors are all possible different combinations of these taken 1, 2, 3, 4 and 5 at a time

Factors of algebraic expressions

List the factors of $x(x + 3)(x - 5)$

$x(x + 3)(x - 5) = 1 \times x \times (x + 3) \times (x - 5)$

The factors are all possible combinations of these:

$1, x, x + 3, x - 5, x(x + 3), x(x - 5), (x + 3)(x - 5), x(x + 3)(x - 5)$

Substitution into algebraic expressions

What is the value of the expression $3x^3 - 5(x - y)$ when $x = 2$ and $y = 5$?

Remember to use the order of operations (BIDMAS).

$3x^3 - 5(x - y)$

$= 3 \times 2^3 - 5 \times (2 - 5)$

$= 3 \times 2^3 - 5 \times -3$ Brackets

$= 3 \times 8 - 5 \times -3$ Indices

$= 24 - (-15)$ Multiplication/Division

$= 24 + 15$ Addition/Subtraction

$= 39$

Substitution into algebraic expressions

What is the value of the expression $6x^2 - 3(y^3 - 2x)$ when $x = 4$ and $y = -2$?

$6x^2 - 3(y^3 - 2x)$

$= 6 \times 4^2 - 3 \times ((-2)^3 - 2 \times 4)$

$= 6 \times 4^2 - 3 \times (-8 - 8)$

$= 6 \times 16 - 3 \times -16$

$= 96 - -48$

$= 96 + 48$

$= 144$

Substitution into formulae

A formula is given as:

$$\frac{B}{2} = \frac{P}{3D\left(D - \sqrt{D^2 - d^2}\right)}$$

What is the value of B if $P = 27$, $D = 17$ and $d = 8$?

$$\frac{B}{2} = \frac{P}{3D\left(D - \sqrt{D^2 - d^2}\right)} = \frac{27}{3\times17\times\left(17 - \sqrt{17^2 - 8^2}\right)} = \frac{27}{3\times17\times\left(17 - \sqrt{225}\right)} = \frac{27}{3\times17\times(17 - 15)} = \frac{27}{3\times17\times2}$$

So

$$B = 2 \times \frac{27}{3\times17\times2} = \frac{27}{3\times17} = \frac{9}{17}$$

Note: we could have used $\sqrt{17^2 - 8^2} = \sqrt{(17-8)(17+8)} = \sqrt{9}\ \sqrt{25} = 3 \times 5 = 15$

M4.4

Collect like terms, multiply a single term over a bracket, take out common factors, and expand products of two or more binomials.

Like terms

Like terms in algebra are identical apart from the numerical constant multiplier which may or may not be the same.

$12x^2y^4$ and $-6x^2y^4$ are like terms. $10y^2$ and $0.75y^2$ are like terms.
$12x^2y^4$ and $12x^2y^3$ are not like terms.
$10y^2$ and $10y$ are not like terms.

Like terms can be collected, they can then be combined by adding and subtracting.

Multiplying a single term over a bracket

A single term can be multiplied over a bracket by multiplying every term inside the bracket by the term outside the bracket:

$a(b + c) = ab + ac$

Taking out common factors

Common factors of a set of terms are factors which appear in every term.

A common factor can be taken out of an expression:

$abc + ab + ac = a(bc + b + c)$

Expanding products of two or more binomials

Brackets containing two or more terms can be multiplied together by multiplying every term in one bracket by every term in the other bracket:

$(x + a)(x + b) = x^2 + ax + bx + ab = x^2 + (a + b)x + ab$

Collecting like terms

Simplify the following expression by collecting like terms:
$3x - 4x^2 + 3y + 3x^3 - 7x + 8x^2 + 5x - y^2$

Collect like terms, remembering that x and x^2 are not like terms.

x terms: $3x - 7x + 5x = +1x = +x$

x^2 terms: $-4x^2 + 8x^2 = +4x^2$

x^3 terms: $+3x^3$

y terms: $+3y$

y^2 terms: $-y^2$

So $3x - 4x^2 + 3y + 3x^3 - 7x + 8x^2 + 5x - y^2 = x + 4x^2 + 3x^3 + 3y - y^2$

This expression is correct as it is, but is often rearranged either in alphabetical order with the highest index first:

$3x^3 + 4x^2 + x - y^2 + 3y$

Or by putting terms with highest indices first:

$3x^3 + 4x^2 - y^2 + x + 3y$

Multiplying a single term over a bracket

Multiply out: $-3p(2p - 5q + 6r)$

Method 1:
multiply every term in the bracket by the term outside the bracket.
$-3p \times 2p = -6p^2$
$-3p \times -5q = +15pq$ $-3p \times +6r = -18pr$
So $-3p(2p - 5q + 6r) = -6p^2 + 15pq - 18pr$

Method 2:
set the calculation out in a grid.

×	$+2p$	$-5q$	$+6r$
$-3p$	$-6p^2$	$+15pq$	$-18pr$

Taking out common factors

Factorise:

$6x^2y^3z + 15x^3y^2z^2 - 21x^5y$

Method 1

Look at the numbers first: 6, 15, 21. The highest common factor is 3.
With the letters, first check that the letter is present in every term.
The letters present in every term are x and y.
Look for the power of each letter that is contained within every term. For x, this is x^2. For y, this is y^1 or just y.

The common factor is $3x^2y$.

Now divide every term in the bracket by the common factor. Divide the numbers and subtract the indices of the letters for each term.

$6x^2y^3z \div 3x^2y = 2x^0y^2z = 2y^2z$

$15x^3y^2z^2 \div 3x^2y = 5xyz^2$

$21x^5y \div 3x^2y = 7x^3y^0 = 7x^3$

$3x^2y(2y^2z + 5xyz^2 - 7x^3)$

Method 2

Write out each term in full and then identify terms which appear in all three. These are the common factors.

$6x^2y^3z = 3 \times 2 \times x \times x \times y \times y \times y \times z = 3x^2y(2 \times y \times y \times y) = 3x^2y(2y^2z)$

$15x^3y^2z^2 = 3 \times 5 \times x \times x \times x \times y \times y \times z \times z = 3x^2y(5 \times x \times y \times z \times z) = 3x^2y(5xyz^2)$

$21x^5y = 3 \times 7 \times x \times x \times x \times x \times x \times y = 3x^2y(7 \times x \times x \times x) = 3x^2y(7x^3)$

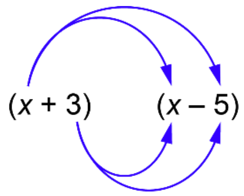
So $6x^2y^3z + 15x^3y^2z^2 - 21x^5y = 3x^2y(2y^2z + 5xyz^2 - 7x^3)$

Expanding products of two binomials

Expand and simplify: $(x + 3)(x - 5)$

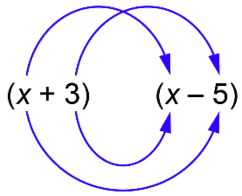
Method 1

Multiply every term in the first bracket by every term in the second bracket.



giving $x^2 - 5x + 3x - 15 = x^2 - 2x - 15$

or



giving $x^2 - 15 - 5x + 3x = x^2 - 2x - 15$

Method 2

Split the multiplication:

$(x + 3)(x - 5) = x(x - 5) + 3(x - 5) = x^2 - 5x + 3x - 15 = x^2 - 2x - 15$

Method 3

Use a grid – probably the safest method, especially when dealing with longer brackets.

×	x	+3
x	x^2	$+3x$
−5	$-5x$	−15

$(x + 3)(x - 5) = x^2 + 3x - 5x - 15 = x^2 - 2x - 15$

Expanding products of more than two binomials

Multiply: $(x + 3)(x - 5)(2x - 5)$

Multiply the first two brackets together as in the previous example:
$(x + 3)(x - 5) = x^2 - 2x - 15$
Now multiply $(2x - 5)(x^2 - 2x - 15)$ by any of the methods in the previous example. Splitting and grid are probably the best methods:

Method 1: Splitting

$(2x - 5)(x^2 - 2x - 15) = 2x(x^2 - 2x - 15) - 5(x\hat{}2 - 2x - 15)$
remember $-5 \times -2 = +10$
$= 2x^3 - 4x^2 - 30x - 5x^2 + 10x + 75$
$= 2x^3 - 9x^2 - 20x + 75$

Method 2: Grid

×	x^2	$-2x$	-15
$2x$	$2x^3$	$-4x^2$	$-30x$
-5	$-5x^2$	$+10x$	$+75$

$(2x - 5)(x^2 - 2x - 15) = 2x^3 - 4x^2 - 5x^2 - 30x + 10x + 75 = 2x^3 - 9x^2 - 20x + 75$

M4.5

Factorise quadratic expressions of the form $x^2 + bx + c$, including the difference of two squares.

Factorise quadratic expressions of the form $ax^2 + bx + c$, including the difference of two squares.

Factorising quadratic expressions of the form $x^2 + bx + c$

Some quadratic expressions of the form $x^2 + bx + c$ can be expressed as a product of two linear expressions with integer coefficients: $(x + p)(x + q)$ where $b = p + q$ and $c = pq$. In other words, p and q are two numbers whose product is c and whose sum is b.

For example, $x^2 + 5x + 6 = (x + 3)(x + 2)$

Factorising quadratic expressions of the form $x^2 - a^2$ - the difference of two squares

Quadratic expressions of the form $x^2 - a^2$ are called the difference of two squares and can be factorised as $(x + a)(x - a)$.

For example, $x^2 - 9 = (x + 3)(x - 3)$

Factorising quadratic expressions of the form $ax^2 + bx + c$

Some quadratic expressions of the form $ax^2 + bx + c$ can be expressed as a product of two linear expressions with integer coefficients: $(px + q)(rx + s)$ where $a = pr, c = qs$ and $b = ps + rq$.

For example, $2x^2 + 7x + 6 = (2x + 3)(x + 2)$

Factorising quadratic expressions of the form $a^2 x^2 - b^2$ – the difference of two squares

Quadratic expressions of the form $a^2 x^2 - b^2$ are a difference of two squares and can be factorised as $(ax + b)(ax - b)$.

For example, $9x^2 - 25 = (3x + 5)(3x - 5)$.

Factorising quadratic expressions of the form $x^2 + bx + c$

Factorise $x^2 + 17x + 72$

To factorise means to express $x^2 + 17x + 72$ as $(x + p)(x + q)$ You need to find two numbers p and q where $p + q = 17$ and $pq = 72$

Two numbers whose product is 72 and which add up to 17 are 8 and 9.

So $x^2 + 17x + 72 = (x + 8)(x + 9)$

Factorising quadratic expressions of the form $x^2 + bx + c$

Factorise $x^2 - x - 72$

$x^2 - x - 72 = x^2 - 1x - 72$

You need two numbers whose product is −72 and whose sum is −1.

The numbers are −9 and +8.

$$x^2 - x - 72 = (x - 9)(x + 8)$$

Factorising quadratic expressions of the form $x^2 - a^2$ – the difference of two squares

Factorise $x^2 - 16$

Method 1

$x^2 - 16 = x^2 + 0x - 16$

You need two numbers whose product is −16 and whose sum is 0. The numbers are +4 and −4.

$x^2 - 16 = (x + 4)(x - 4)$

Method 2

Use the difference of two squares, noting 16 $=4^2$

$x^2 - 16 == (x + 4)(x - 4)$

Factorising quadratic expressions of the form x² – a² – the difference of two squares

Factorise $x^2 - 3$

$x^2 - 3 = (\sqrt{(x^2)} + \sqrt{3})(\sqrt{(x^2)} - \sqrt{3}) = (x + \sqrt{3})(x - \sqrt{3})$

Factorising quadratic expressions of the form $ax^2 + bx + c$

Factorise $6x^2 + 23x + 20$

You need to put this into two brackets which multiply together to give $6x^2 + 23x + 20$

Method 1

If $6x^2 + 23x + 20 = (px + q)(rx + s)$

Expanding the bracket on the right-hand side, we see:

$$(px + q)(rx + s) = prx^2 + (ps + qr)x + qs$$

So $p \times r = 6$ and $q \times s = 20$ *and* $p \times s + q \times r = 23$.

Start by considering pairs of values which will give $p \times r = 6$ and $q \times s = 20$.

p	r
6	1
1	6
2	3
3	2

q	s
1	20
20	1
2	10
10	2
4	5
5	4

Note: Negative values would also be possible, but we do not need to consider these in this case because all of the terms in the original quadratic are positive.

Now we identify which combination of p, q, r, s will give $p \times s + q \times r = 23$.

	$q = 1, s = 20$	$q = 20, s = 1$	$q = 2, s = 10$	$q = 10, s = 2$	$q = 4, s = 5$	$q = 5, s = 4$
$p = 6, r = 1$	121	26	62	22	34	29
$p = 1, r = 6$	26	121	22	62	29	34
$p = 2, r = 3$	43	62	26	34	22	23
$p = 3, r = 2$	62	43	34	26	23	22

Note: The first and second rows and the third and fourth rows have the same numbers present. This is because the pairings generate the same potential factorisations but with the brackets in the two possible orders.

You do not have to draw a table to find this – you may be able to identify the correct pairs without showing all of this working.

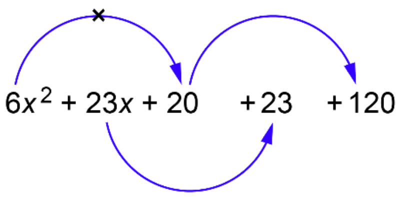
$p = 2, r = 3, q = 5, s = 4$

$(px + q)(rx + s) = (2x + 5)(3x + 4)$

Method 2

$6x^2 + 23x + 20$

You need to find two numbers which multiply to give +120 (x^2 coefficient multiplied by the constant term, i.e. 6 × 20) and add to give +23 (coefficient of x term):



These numbers are +15 and +8.

Split the $+23x$ into $+15x$ and $+8x$ and then take out common factors for the first two terms and the last two terms.

$6x^2 + 23x + 20 = 6x^2 + 15x + 8x + 20$

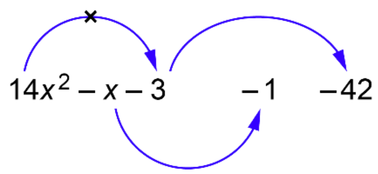
$= 3x(2x + 5) + 4(2x + 5)$ which have a common factor of $(2x + 5)$

$= (2x + 5)(3x + 4)$

Factorising quadratic expressions of the form $ax^2 + bx + c$

Factorise $14x^2 - x - 3$

Two numbers which multiply to −42 and add to −1 are −7 and +6:



Split the middle term and factorise, remembering that a minus sign outside the bracket changes the sign inside the bracket.

$14x^2 - x - 3 = 14x^2 + 6x - 7x - 3 = 2x(7x + 3) - 1(7x + 3) = (7x + 3)(2x - 1)$

Factorising quadratic expressions of the form $ax^2 - b^2$ – the difference of two squares

Factorise $16x^2 - 49$

$16x^2 - 49 = (\sqrt{16x^2} + \sqrt{49})(\sqrt{16x^2} - \sqrt{49}) = (4x + 7)(4x - 7)$

M4.6

Simplify expressions involving sums, products and powers, including the laws of indices.

Simplify rational expressions by cancelling, or factorising and cancelling.

Use the four rules on algebraic rational expressions.

Simplifying expressions involving sums, products and powers

Expressions can be simplified in a number of ways.

The usual method is:

- simplify any terms involving fractions by cancelling,
- if possible collect like terms if there are any
- look for common factors of two or more terms
- if there are no common factors, then multiply out any brackets and repeat the process - simplify fractions
- collect like terms

Cancelling

Rational expressions can be simplified in the same way as numerical fractions by dividing both the numerator and denominator by the same number, term or expression until no further division is possible.

Rational expressions involving sums or differences

If the rational expression involves sums or differences, then factorise the sum or difference before proceeding.

Rational expressions involving quadratics

If the rational expression involves quadratic expressions, then factorise those expressions before trying to simplify.

Adding and subtracting rational expressions

Algebraic rational expressions can be added or subtracted by putting the expressions over a common denominator using the same rules as apply to numerical fractions.

The most useful common denominator is the lowest common multiple (LCM) of the two denominators but a denominator involving the product of the denominators can be used.

In general: $\frac{x}{a} + \frac{y}{b} = \frac{xb+ya}{ab}$

Note:

The LCM of 6x and 15y is 30xy but the common multiple of 90xy could be used.
The LCM of $6x^4$ and $15x^2y$ is $30x^4y$ but the common multiple of $90x^6y$ could be used.
The advantage of using the LCM is that it reduces the need to cancel in the final expression.

Multiplying and dividing rational expressions

Algebraic rational expressions can be multiplied and divided using the same rules as apply to numerical fractions.

In general:

$\frac{x}{a} \times \frac{y}{b} = \frac{xy}{ab}$

$\frac{x}{a} \div \frac{y}{b} = \frac{x}{a} \times \frac{b}{y} = \frac{xb}{ay}$

Simplifying expressions involving sums, products and powers

Simplify:

$3x^2 y + 4x + 9x(xy + 2) + 2x$

Multiplying out the bracket and simplifying the fraction gives:

$3x^2 y + 4x + 9x^2 y + 18x + 2x$

$= 12x^2 y + 24x$ (collecting like terms)

$= 12x(xy + 2)$ (taking out a common factor)

Simplify:

$3x + 20y + 12 + 5xy$

In this case there are no common factors so rearrange the terms and try to factorise the terms in pairs again.

$3x + 20y + 12 + 5xy$
$= 20y + 5xy + 3x + 12$ (rearranging the terms)
$= 5y(4 + x) + 3(x + 4)$ (factorise the terms in pairs)
$= 5y(x + 4) + 3(x + 4)$ (rearrange the first bracket)
$= (x + 4)(5y + 3)$ ($(x + 4)$ is a common factor)

Cancelling

Simplify

$(6x^3 y^4)/(15x^5 y)$

$= \frac{2x^3 y^4}{5x^5 y}$ (divide numerator and denominator by 3)

$= \frac{2y^4}{5x^2 y}$ (divide numerator and denominator by x^3)

$= \frac{2y^3}{5x^2}$ (divide numerator and denominator by y)

Rational expressions involving sums or differences

Simplify

$\frac{4x^2 y+8xy^2}{3x+6y}$

Take out common factors of numerator and denominator:

$\frac{4x^2 y+8xy^2}{3x+6y} = \frac{4xy(x+2y)}{3(x+2y)}$

Cancel terms:

$\frac{4xy(x+2y)}{3(x+2y)} = \frac{4xy}{3}$

Rational expressions involving quadratics

Simplify

$\frac{6x^2+19x+10}{4x^2-25}$

Factorise numerator and denominator (difference of 2 squares):

$\frac{6x^2+19x+10}{4x^2-25} = \frac{(3x+2\ )(2x+5)}{(2x+5)(2x-5)}$

Cancel terms

$\frac{(3x+2)(2x+5)}{(2x+5)(2x-5)} = \frac{(3x+2)}{(2x-5)}$

Adding and subtracting rational expressions

Simplify

$$\frac{p+1}{p-p^2} - \frac{2}{p-1}$$

Factorise any numerator or denominator which will factorise

$$\frac{p+1}{p-p^2} - \frac{2}{p-1} = \frac{p+1}{p(1-p)} - \frac{2}{p-1}$$

Multiply top and bottom of second fraction by -1

$$\frac{p+1}{p(1-p)} - \frac{2}{p-1} = \frac{p+1}{p(1-p)} + \frac{2}{1-p}$$

Put terms over a common denominator

$$\frac{p+1}{p(1-p)} + \frac{2}{1-p} = \frac{(p+1)+2p}{p(1-p)}$$

Simplify

$$\frac{(p+1)+2p}{p(1-p)} = \frac{3p+1}{p(1-p)}$$

M4.7

Rearrange formulae to change the subject.

Changing the subject of the formula means expressing one variable in the formula in terms of the other variables. To do this the formula must be rearranged according to the rules of arithmetic and algebra in order to isolate the new subject:

- You can add or subtract the same quantity or term from each side of a formula.
- You can multiply or divide both sides of a formula by the same non-zero quantity.
- You can invert both sides of a formula if both sides are a single fraction with non-zero denominators.
- You can square both sides of a formula or raise both sides of a formula to the same non-zero power.
- You can take the square root of both sides of a formula.

Changing the subject of a formula using subtraction

Make p the subject of the formula:

$q + 2p = p + 4q - 6r$

$q + 2p - p = 4q - 6r$ Subtract p from both sides to get all the terms in p on the same side

$q + p = 4q - 6r$ Collect like terms

$p = 4q - 6r - q$ Subtract q from both sides to isolate p

$p = 3q - 6r$ Collect like terms

$p = 3(q - 2r)$ Take out the common factor of 3

Changing the subject of a formula using the four functions

Make x the subject of the formula:

$y = 3(x - 2) + \frac{x}{2}$

$y = 3x - 6 + \frac{x}{2}$ Multiply out the bracket

$2y = 6x - 12 + x$ Multiply both sides of the formula by 2 to get rid of the fraction

$2y = 7x - 12$ Collect like terms

$2y + 12 = 7x$ Add 12 to both sides

$x = \frac{2y+12}{7}$ Divide by 7

Changing the subject of the formula where the variable to be the subject is in the denominator of a fraction

Make u the subject of the formula:

$\frac{1}{v} - \frac{1}{u} = \frac{1}{p}$

$\frac{1}{v} - \frac{1}{p} = \frac{1}{u}$ Rearrange to get $\frac{1}{u}$ by itself

$\frac{1}{v} - \frac{1}{p} = \frac{p-v}{vp} = \frac{1}{u}$ Put left hand side over a common denominator

$\frac{vp}{p-v} = \frac{u}{1} = u$ Invert both fractions [turn them both upside down]

Changing the subject of a formula involving a square root

Make x the subject of the formula:

$y = \sqrt{\frac{3x+5}{4}}$

Square both sides

$y^2 = \frac{3x+5}{4}$

Then rearrange:

$4y^2 = 3x + 5$

$4y^2 - 5 = 3x$

$x = \frac{(4y^2-5)}{3}$

M4.8

Understand the difference between an equation and an identity.

Argue mathematically to show that algebraic expressions are equivalent.

Definitions

An equation is true only for certain values of a variable, it can be solved for particular values of the variable. $p^2 = 9$ is an equation which is true for $p = \pm 3$

An identity is true for all values of the variable. If an attempt is made to solve an identity the result will be 0 = 0, which is true but unhelpful. $5x + 3 = 3x + 2 + 2x + 1$ is an identity.

Showing two algebraic expressions are equivalent

To show that two algebraic expressions are equivalent you need to show that they both reduce to exactly the same form.

Equation or identity?

Is $\frac{4}{x+1} - \frac{3}{x-1} = \frac{1}{x^2-1}$ an equation or an identity?

Simplify the left-hand side (LHS) and equate to the right-hand side (RHS). If they are identical, it is an identity; if not, it is an equation:

$$\frac{4}{x+1} - \frac{3}{x-1} = \frac{4(x-1) - 3(x+1)}{(x+1)(x-1)} = \frac{4x - 4 - 3x - 3}{x^2 - 1} = \frac{x-7}{x^2-1}$$

But this is not identical to the RHS so this is not an identity.

It is an equation which is true for $x - 7 = 1,\ x = 8$

Arguing mathematically to show algebraic expressions are equivalent

Show that

$$\frac{2p}{3} + \frac{3q}{2} - \frac{5(p+q)}{6} = \frac{4q-p}{66}$$

We start by simplifying the LHS by putting it over a common denominator of 6:

$$\frac{2p}{3} + \frac{3q}{2} - \frac{5(p+q)}{6} = \frac{4p+9q-5(p+q)}{6}$$

And then simplifying:

$$\frac{4p+9q-5(p+q)}{6} = \frac{4p+9q-5p-5q}{6} = \frac{-p+4q}{6} = \frac{4q-p}{6}$$

And this is the RHS as required

M4.9

Work with coordinates in all four quadrants.

The coordinate axes

The coordinate axes x and y are a way of locating points in the plane.
They cross at the origin.

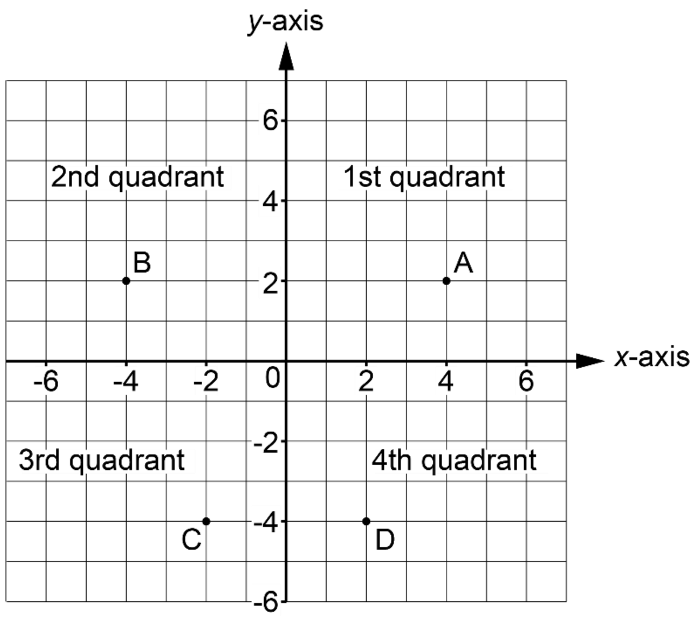
The axes divide the plane into 4 quadrants as shown in the diagram.

The x-axis is usually horizontal, positive numbers to the right of the origin and negative to the left.

The y-axis is usually vertical, positive numbers above the origin and negative numbers below.

The coordinates of a point are given as 2 numbers in a bracket, separated by a comma with the x coordinate first.

A is the point (4,2), B is (−4,2), C is (−2,−4) and D is (2,−4).



The coordinate axes

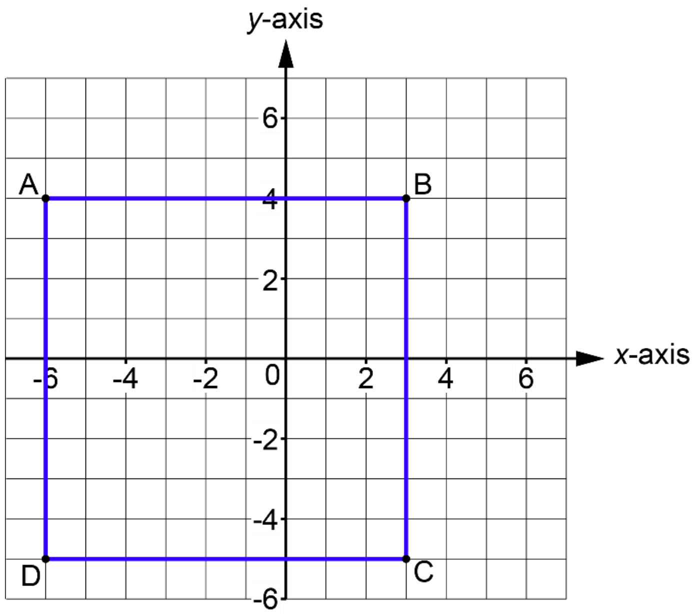
A square ABCD is drawn. A is the point (−6,4), B is the point (3,4). C lies in the 4th quadrant and D lies in the 3rd quadrant.

What are the coordinates of C and D?

A is 6 units to the left of the origin and 4 units up. B is 3 units to the right of the origin and 4 units up. The length of the side of the square is 9 units.

If C is in the fourth quadrant it is 9 units below B so it is at (3,−5).

If D is in the fourth quadrant it is 9 units below A so it is at (−6,−5).



M4.10

Identify and interpret gradients and intercepts of linear functions ($y = mx + c$) graphically and algebraically.

Identify pairs of parallel lines and identify pairs of perpendicular lines, including the relationships between gradients.

Find the equation of the line through two given points, or through one point with a given gradient.

Equation of a straight line

The equation of a straight line can be written in the form $y = mx + c$ where m is the gradient of the line and c is the intercept with the y–axis.

Parallel lines

Parallel lines have the same gradient.

Perpendicular lines

If two lines are perpendicular, then the product of their gradients is −1.

Equation of a line given the gradient and a point on the line

The equation of the line with gradient m which goes through the point (p,q) is $y = mx + c$ where $c = q - mp$

Equation of a line joining two given points

The gradient of the line joining point (x,y) and (x_1,y_1) is

$$\frac{y-y_1}{x-x_1} = \frac{difference\ in\ y}{difference\ in\ x}$$

With this gradient, and either of the two points, the equation of the line can be found.

Equation of a straight line

What is the gradient of the line $y - 3x = 6$ and where does the line intersect the y-axis?

Put the equation into the form $y = mx + c$ where m is the gradient of the line and c is the intercept with the y-axis.

$$y - 3x = 6$$

Adding $3x$ to both sides gives $y = 3x + 6$

Comparing this with $y = mx + c$ gives $m = 3$ and $c = 6$ so the gradient of the line is 3 and the line cuts the y–axis where $y = 6$

What is the gradient of the line $2y - 3x = 6$ and where does the line intersect the y-axis?

Put the equation into the form $y = mx + c$ and c is the intercept with the y–axis.

$$2y - 3x = 6$$

Adding $3x$ to both sides gives $2y = 3x + 6$

Dividing both sides by 2 gives $y = \frac{3}{2}x + 3$

Comparing this with $y = mx + c$ gives $m = \frac{3}{2}$ and $c = 3$ so the gradient of the line is $\frac{3}{2}$ and the line cuts the y–axis where $y = 3$

Parallel lines

Which of these lines are parallel?

a: $y + 4x = 9$
b: $y = 4x + 7$
c: $3y + 4x = 9$
d: $2y = 7 - 8x$
e: $3y - 12x = -7$

If lines are parallel their gradients are the same so find the gradients of all the lines by putting them in the form $y = mx + c$ where m is the gradient of the line.

Rearrange $y + 4x = 9$ as $y = -4x + 9$ which has gradient -4

$y = 4x + 7$ has gradient $+4$

Rearrange $3y + 4x = 9$ as $y = -\frac{4}{3}x + 3$ which has gradient $-\frac{4}{3}$

Rearrange $2y = 7 - 8x$ as $y = -4x + \frac{7}{2}$ which has gradient -4

Rearrange $3y - 12x = -7$ as $y = 4x - \frac{7}{3}$ which has gradient +4

So a and d are parallel as they both have gradient -4; b and e are parallel as they both have gradient $+4$

Perpendicular lines

Which of these lines are perpendicular?

a: $y + 4x = 9$
b: $y = 4x + 7$
c: $4y + 3x = 9$
d: $4y = 7 - x$
e: $3y - 4x = -7$

If lines are perpendicular, the product of their gradients is -1, so find the gradients of all the lines by putting them in the form $y = mx + c$ where m is the gradient of the line.

Rearrange $y + 4x = 9$ as $y = -4x + 9$ which has gradient -4

$y = 4x + 7$ has gradient $+4$

Rearrange $4y + 3x = 9$ as $y = -\frac{3}{4}x + \frac{9}{4}$ which has gradient $-\frac{3}{4}$

Rearrange $4y = 7 - x$ as $y = -\frac{1}{4}x + \frac{7}{4}$ which has gradient $-\frac{1}{4}$

Rearrange $3y - 4x = -7$ as $y = \frac{4}{3}x - \frac{7}{3}$ which has gradient $\frac{4}{3}$

The pairs where the product of the gradients is -1 are lines b and d and lines c and e.

Equation of a line given the gradient and a point on the line

What is the equation of the straight line with gradient -5 through the point $(1, -2)$?

The equation of a straight line is $y = mx + c$ where m is the gradient.

If $m = -5$ then $y = -5x + c$ (i)

The line goes through the point $(1, -2)$ so, substituting in (i):

$$-2 = -5 \times 1 + c \therefore c = 3$$

So $y = -5x + 3$ which can be rearranged to give $y + 5x = 3$

Equation of a line joining two given points

What is the equation of the straight line joining the points (3,5) and $(-1, -3)$?

The gradient of the line is $\frac{difference\ in\ y}{difference\ in\ x} = \frac{5--3}{3--1} = \frac{8}{4} = 2$

The equation of the line is $y = 2x + c$ (i)

To find c substitute either (3,5) or $(-1, -3)$ into (i)

Using (3,5): $5 = 2 \times 3 + c$ so $c = 5 - 6 = -1$

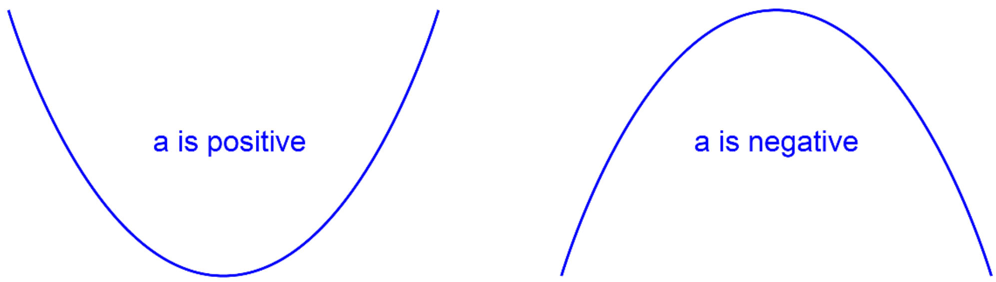
The equation is $y = 2x - 1$

M4.11

Identify and interpret roots, intercepts and turning points of quadratic functions graphically.

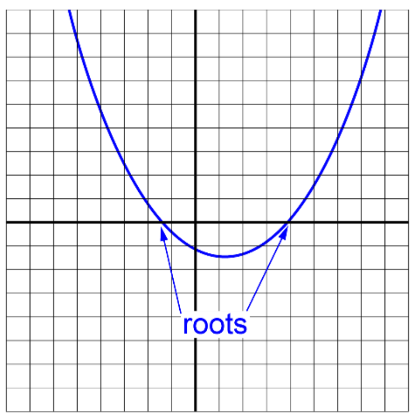
Deduce roots algebraically, and turning points by completing the square.

A quadratic function of a variable x is a function of the form $f(x) = ax^2 + bx + c$ where a, b, and c are constants. The graph of $f(x)$ is a parabola and its orientation depends on the value of a.



Roots of quadratic functions graphically

If the graph of a quadratic function $f(x) = ax^2 + bx + c$ is plotted, then the points where the graph intercepts the xaxis are the roots of the function, and the solutions of the equation $ax^2 + bx + c = 0$

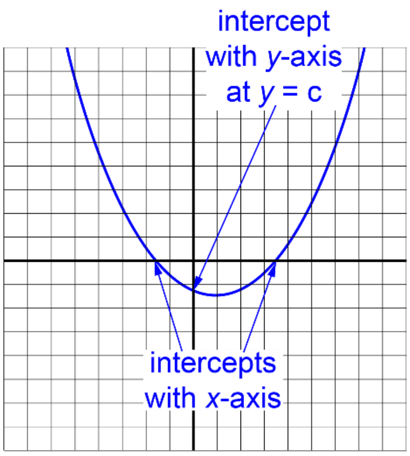


Intercepts of quadratic functions

The intercept of the graph $y = ax^2 + bx + c$ with the y-axis occurs when $x = 0$ so $y = c$. This can be read from the graph at the point where the curve crosses the y-axis.

The intercept of the graph $y = ax^2 + bx + c$ with the x-axis occurs when $y = 0$ so when $ax^2 + bx + c = 0$

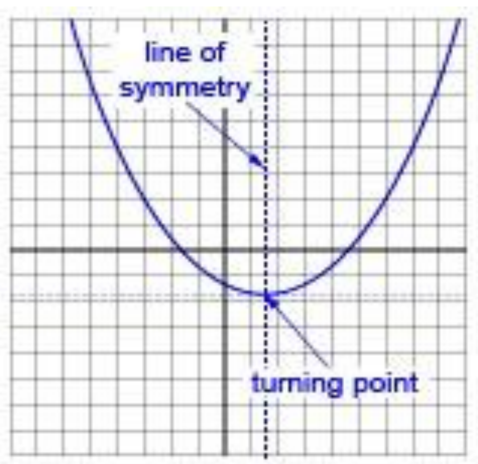
There can be 0, 1 or 2 intercepts with the x-axis. These can be read from the graph at the point(s) where the curve crosses the x-axis.



The turning point of quadratic functions

The turning point of a quadratic graph is the point where the graph is lowest if it is a U-shaped parabola (i.e. $a > 0$ in $y = ax^2 + bx + c$) or the point where the graph is highest if it is an upside-down U shape (i.e. $a < 0$ in $y = ax^2 + bx + c$).

It is also useful to note that a quadratic graph is always symmetric. The line of symmetry goes through the turning point:



Finding the roots of a quadratic function algebraically

If $f(x) = ax^2 + bx + c$ then the roots can be found algebraically by solving the equation $ax^2 + bx + c = 0$

Completing the square

$\left(x + \frac{k}{2}\right)^2 = x^2 + kx + \left(\frac{k}{2}\right)^2$ rearranges to $x^2 + kx = \left(x + \frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2$

where k is a constant.

This process of writing $x^2 + kx$ as a difference of two squares is called completing the square.

Finding the turning point by completing the square

Recall, the turning point is the point on the curve with the lowest y value if $a > 0$ or the point with the highest y value if $a < 0$

Here is the theory, although it is easier to try some examples with numbers in.

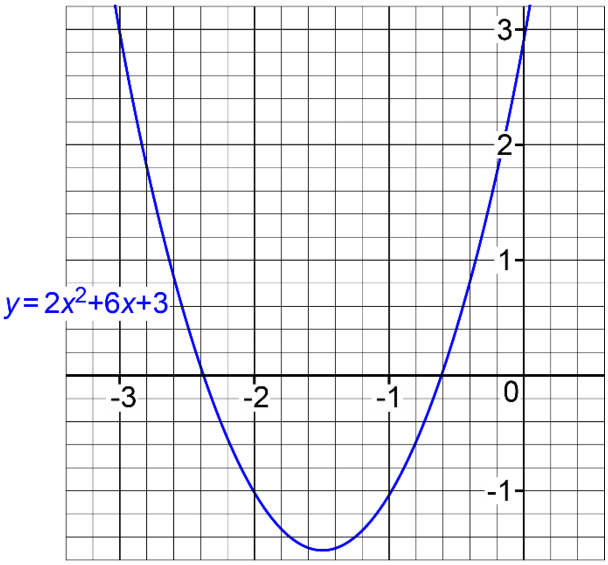
If $y = ax^2 + bx + c$ then $\frac{y}{a} = x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$ = (i)

As a, b, and c are constants they do not change in value, so to make the expression (i) as large [or as small] as possible we require

$x + \frac{b}{2a} = 0$ so $x = -\frac{b}{2a}$ and $y = -\frac{b^2}{4a^2} + \frac{c}{a}$

Roots of quadratic functions graphically

The diagram shows a section of the graph of $y = 2x^2 + 6x + 3$



Give the value of the roots of this function correct to 1 d.p.

The graph is drawn and the roots are the points at which the curve crosses the x-axis.

As 5 small squares make 1 unit then each small square is $1/5$ = 0.2

The roots are at –0.6 and –2.4

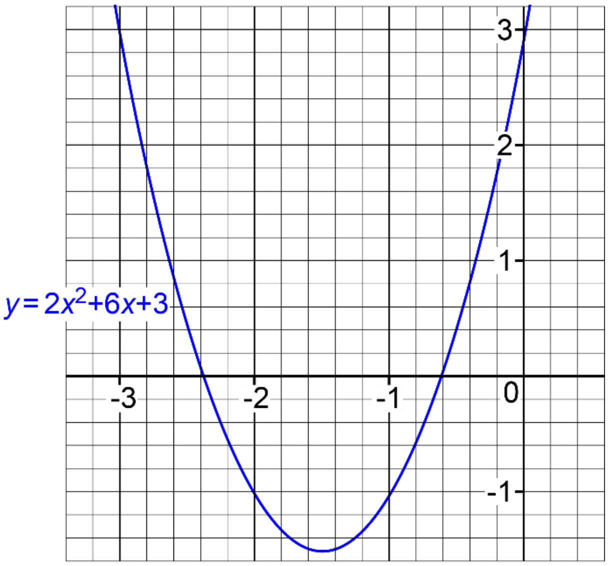
Intercepts of quadratic functions

The graph of $y = 3x^2 + 2x - 5$ is drawn.
What is the intercept of this curve with the y-axis?

The curve crosses the y-axis when $x = 0$
When $x = 0, y = 0 + 0 - 5 = -5$

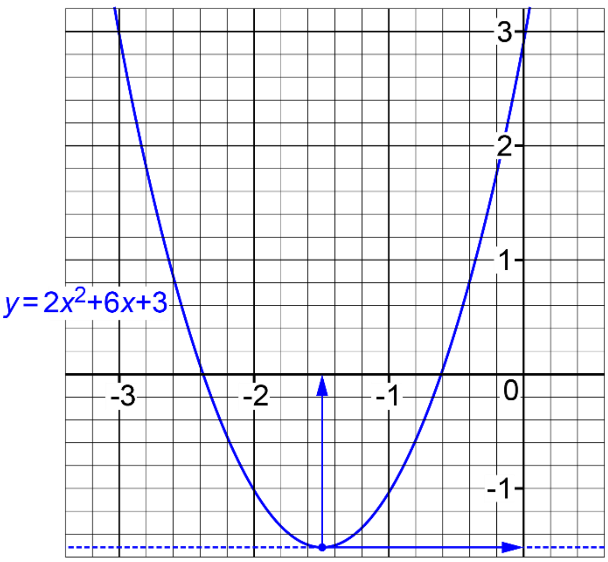
The turning point of quadratic functions

The diagram shows a section of the graph of $y = 2x^2 + 6x + 3$



What are the coordinates of the turning point of the curve correct to 1 d.p.?

The turning point is where the curve is parallel to the x-axis and is at the point $(-1.5, -1.5)$



Finding the roots of a quadratic function algebraically

$f(x) = 2x^2 + 5x + 3$

What are the roots of $f(x)$?

To find the roots of $f(x) = 2x^2 + 5x + 3$ find the solutions of the equation $2x^2 + 5x + 3 = 0$

$2x^2 + 5x + 3 = (2x + 3)(x + 1)$

If $(2x + 3)(x + 1) = 0$ then $2x + 3 = 0$ or $x + 1 = 0$ $x = -\frac{3}{2}$ or $x = -1$

Completing the square

Complete the square:

$\boldsymbol{x^2 + 4x}$

Check that the coefficient of x^2 is 1 and then halve the coefficient of x to get 2.

$(x + 2)^2 = x^2 + 4x + 22$ so $x^2 + 4x = (x + 2)^2 - 2^2$

$\boldsymbol{x^2 - 6x}$

$-6 \div 2 = -3$ so $x^2 - 6x = (x - 3)^2 - (-3)^2 = (x - 3)^2 - 3^2$

$\boldsymbol{2x^2 + 6x}$

$= 2(x^2 + 3x)$ (first take out a factor of 2 so that the coefficient of x^2 is 1)

$= 2[\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2]$ (now complete the square for $x^2 + 3x$)

$= 2\left(x + \frac{3}{2}\right)^2 - 2 \times \left(\frac{3}{2}\right)^2 = 2\left(x + \frac{3}{2}\right)^2 - \frac{9}{2}$

Finding the turning point by completing the square

Find the turning points of these functions by completing the square:

$\boldsymbol{y = x^2 + 6x + 7}$

Complete the square

$x^2 + 6x + 7 = (x+3)^2 - 9 + 7 = (x+3)^2 - 2$

The turning point happens when the bracket containing x is zero so $x + 3 = 0$

This gives $x = -3$ and $y = -2$

$\boldsymbol{y = 2x^2 + 6x + 5}$

Complete the square

$2x^2 + 6x + 5 = 2(x^2 + 3x) + 5 = 2[\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 \ ] + 5 = 2\left(x + \frac{3}{2}\right)^2 - 2 \times \left(\frac{3}{2}\right)^2 + 5$
$= 2\left(x + \frac{3}{2}\right)^2 - \frac{9}{2} + 5 = 2\left(x + \frac{3}{2}\right)^2 + \frac{1}{2}$

This takes its smallest value when $x + \frac{3}{2} = 0$ so $x = -\frac{3}{2}$ and $y = \frac{1}{2}$

M4.12

Recognise, sketch and interpret graphs of:

a. linear functions
b. quadratic functions
c. simple cubic functions
d. the reciprocal function: $y = \frac{1}{x}$ with $x \neq 0$
e. the exponential function: $y = k^x$ for positive values of k
f. trigonometric functions (with arguments in degrees): $y = sin\ x$, $y = cos\ x$, $y = tan\ x$ for angles of any size

Recognise and sketch graphs of:

Linear functions

The graph of a linear function is a straight line.

It can be written in the form $y = mx + c$ where m is the gradient of the line and c is the intercept of the line with the y-axis.

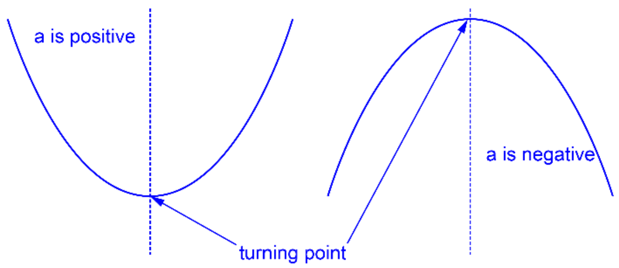
Quadratic functions

The graph of a quadratic function forms a 'u-shaped' curve.

It can be written in the form $y = ax^2 + bx + c$ where c is the intercept of the curve with the y-axis.

If $a > 0$ then the curve is ∪ shaped and if $a < 0$ then the curve is ∩ shaped.

It has a line of symmetry and a single turning point.

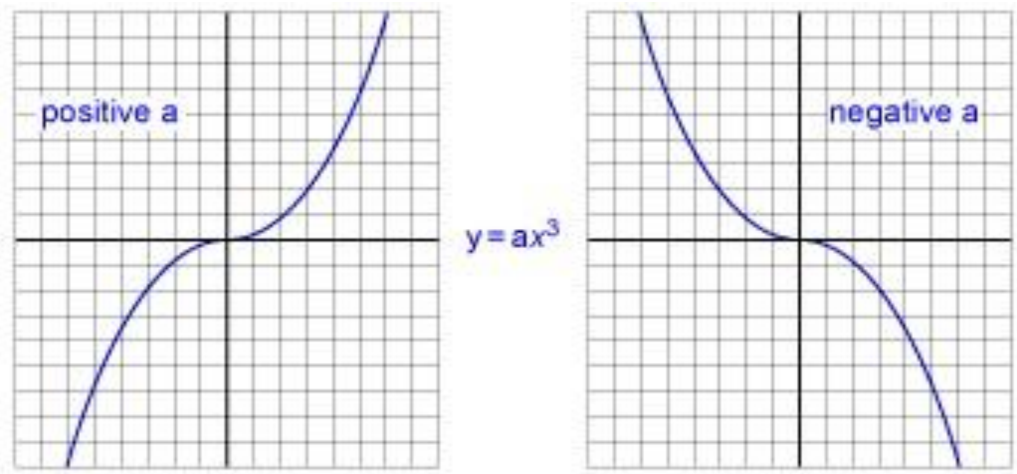


The turning point can be calculated by completing the square.

The intercept(s) of the curve with the x-axis are the roots of the equation $ax^2 + bx + c = 0$

Simple cubic functions

The graph of a basic cubic function of the form $y = ax^3$ has rotational symmetry of order 2 about the origin.



Reciprocal function

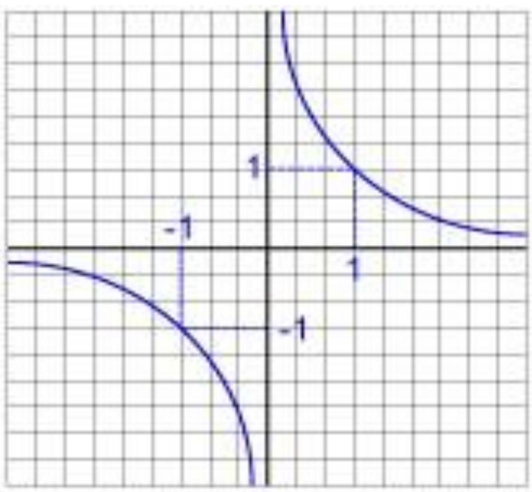
The reciprocal function is $y = \frac{1}{x}$ for $x \neq 0$

If x is positive, then $\frac{1}{x}$ is positive and if x is negative then $\frac{1}{x}$ is negative so the curve exists in the first and third quadrants only.

It goes through the points $(1,1)$ and $(-1,-1)$ and does not touch or cut the axes.

As x gets larger, $y = \frac{1}{x}$ gets smaller.

As x gets smaller, $y = \frac{1}{x}$ gets larger.



The exponential function

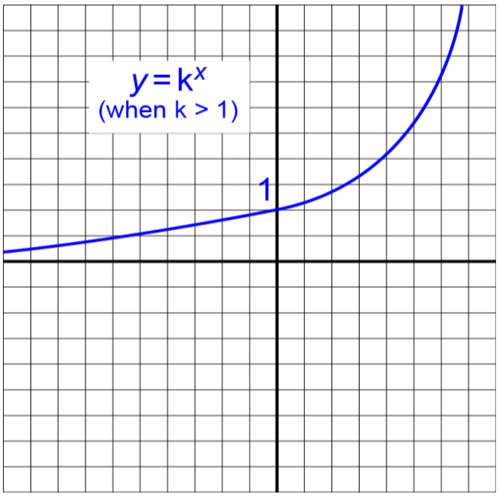
The exponential function is $y = k^x$

All exponential functions go through the point $(0,1)$ as $k^0 = 1$ for all values of k.

When $k > 1$:

As x increases, k^x increases rapidly.

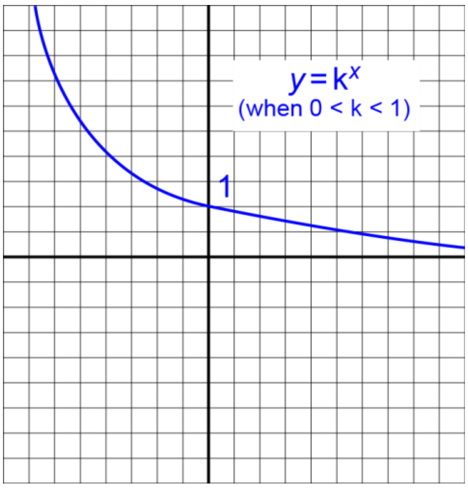
If x is negative then $0 < k^x < 1$ and k^x gets smaller as x gets more negative.



When $0 < k < 1$:

As x increases, k^x decreases rapidly.

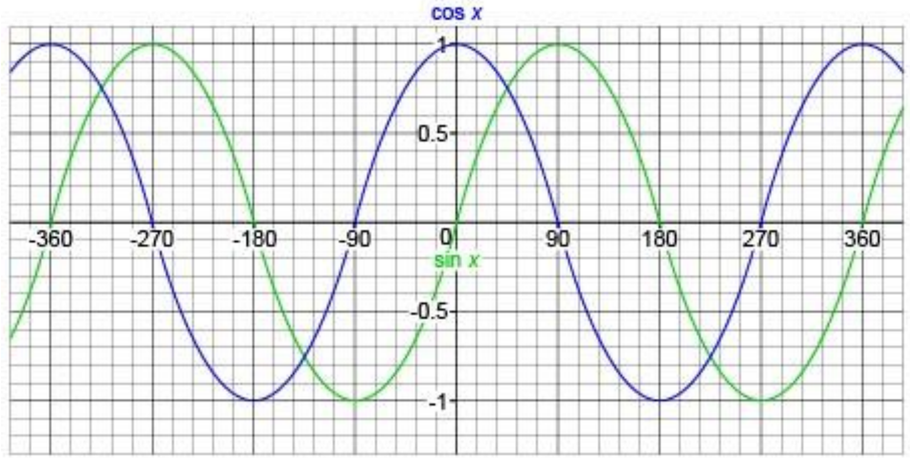
If x is negative, k^x gets larger as x gets more negative.



Trigonometric functions

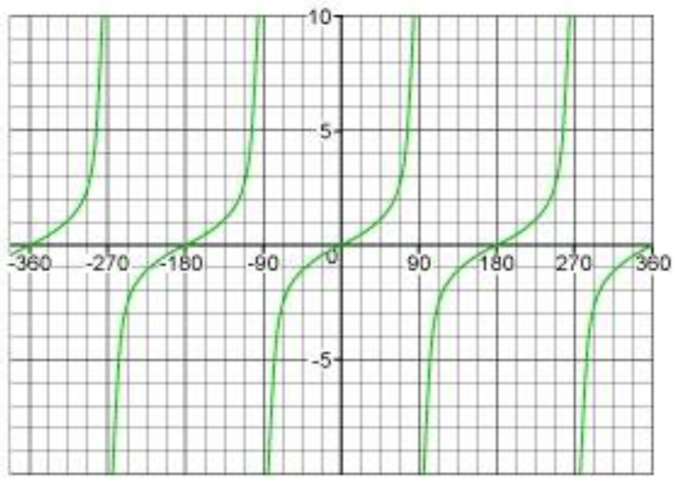
sin θ and cos θ both vary between $+1$ and -1 Key values are:

$\sin 0° = 0$
$\sin 90° = 1$
$\cos 0° = 1$
$\cos 90° = 0$



$\tan \theta$ varies between $-\infty$ and $+\infty$

$\tan 0° = 0$



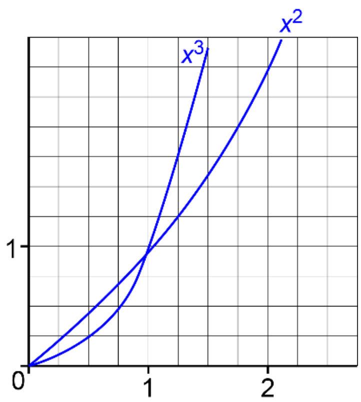
Recognise and sketch graphs

Sketch the graphs of $y = x^2$ and $y = x^3$ for $0 \leq x \leq 2$

When $x = 0, x^2 = 0$ and $x^3 = 0$ When $x = 1, x^2 = 1$ and $x^3 = 1$

For numbers greater than $1, x^3 > x^2$

For numbers between 0 and 1, such as $\frac{1}{2}$, $x^3 < x^2$

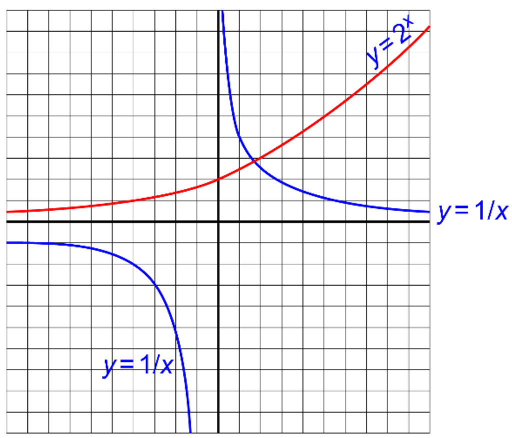


Recognise and sketch graphs

At how many points do the graphs $y = \frac{1}{x}$ and $y = 2^x$ touch or cross?

Sketch the two curves on the same axes.

There is 1 intersection point.



Recognise and sketch graphs

Find, by sketching the graphs, the range of values for which $\sin\theta < \cos\theta$ for $0 \le \theta \le 180°$

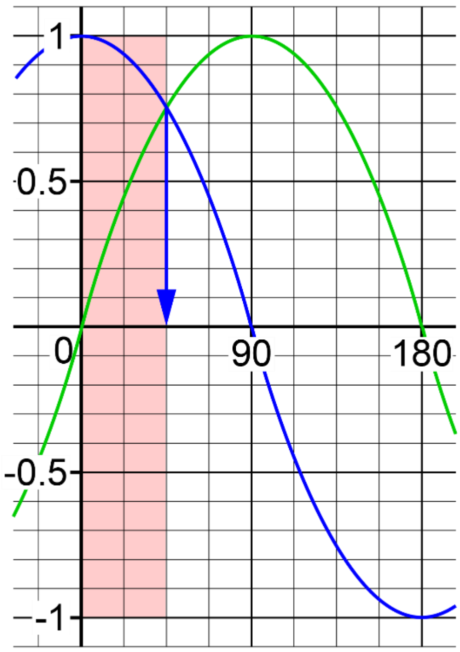
Sketch the graphs of $\sin\theta$ and $\cos\theta$ for $0 \le \theta \le 180°$

At $\theta = 0$, $\sin\theta = 0$ and $\cos\theta = 1$

In the shaded region, $\sin\theta < \cos\theta$ as the graph of $\sin\theta$ is below the graph of $\cos\theta$ θ

The two graphs intersect when $\sin\theta = \cos\theta$ and $\sin\theta = \cos\theta$ when $\theta = 45°$

The range of values for which $\sin\theta < \cos\theta$ is $0 \le \theta \le 45°$



M4.13

Interpret graphs (including reciprocal graphs and exponential graphs) and graphs of non-standard functions in real contexts to find approximate solutions to problems, such as simple kinematic problems involving distance, speed and acceleration.

Straight line graphs

Straight line graphs that pass through the origin represent simple proportional relationships, e.g. cost of items with no reduction for bulk buying, distance travelled at constant speed, exchange rates with no administration fee.

If the line does not go through the origin, then it suggests an initial charge then a proportional relationship, e.g. the initial cost of a mobile phone plus a fixed monthly fee.

Reciprocal graphs

Reciprocal graphs of the form $y = \frac{k}{x}$ are often used to model real life situations. These often show proportion of the form 'y is inversely proportional to x'.

Exponential curves

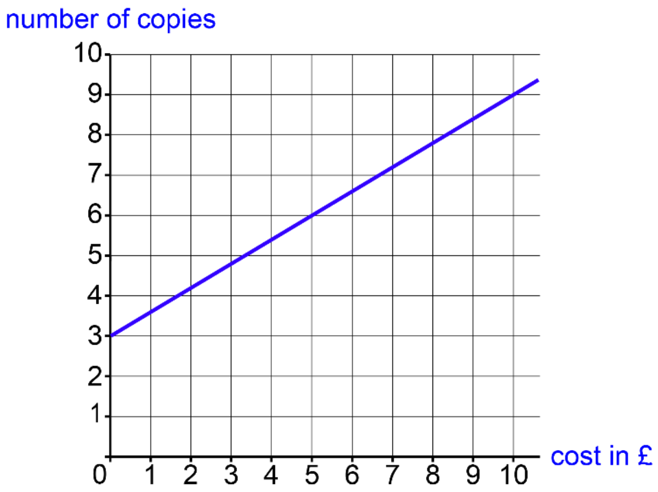
Exponential curves of the form $y = k^x$ are used to model exponential growth or decay, where a quantity increases by the same factor in each period of time, e.g. a population which doubles every year.

Non-standard graphs

Non-standard graphs can either be curves or straight lines. The most common are travel graphs, which have different lines and curves joined to represent different stages of a journey.

Straight line graphs

The cost of printing a poster consists of an initial set-up fee plus a fixed cost for each poster printed.



Find, from the graph, the set-up cost.

Calculate the cost of printing 500 posters.

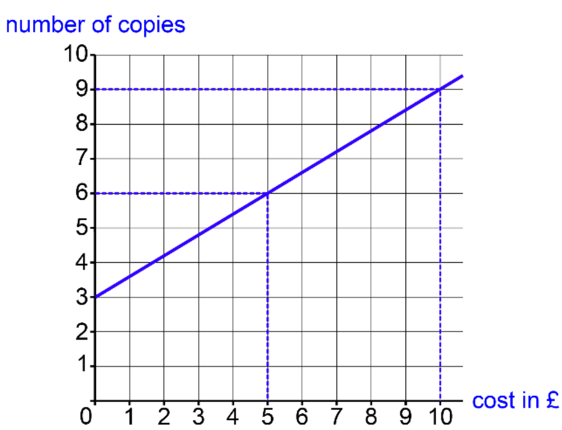
The set-up cost is the cost of 0 copies which, from the graph, is £3.

To calculate the cost, per poster, of printing, find a point that the graph passes through which is easy to read. In this case, (10, 9) is easily identifiable.

To calculate the unit cost using (10, 9):

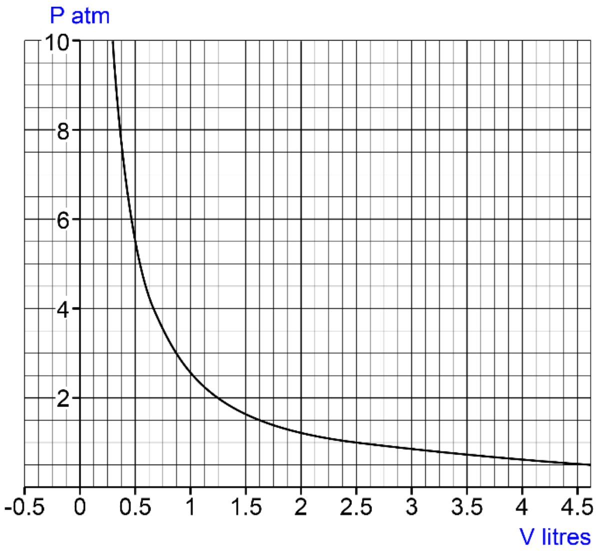
The cost of printing 1 copy is £ $\frac{9-3}{10}$ = £0.60

The cost of 500 copies is £3 + £(500 × 0.60) = £303



Reciprocal graphs

The graph shows the pressure in atmospheres, P, plotted against the volume, V , of a gas in litres.



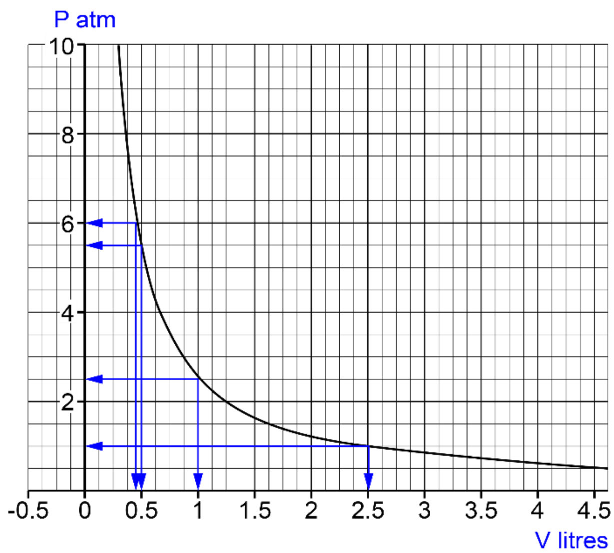
Is it reasonable to deduce from the graph that V is inversely proportional to P?

If P and V are inversely proportional, state the equation connecting P and V .

If V is inversely proportional to P, then $V = \frac{k}{P}$

To find k, check the values for P = 1

From the graph, when P = 1, V = 2.5 so k = 2.5



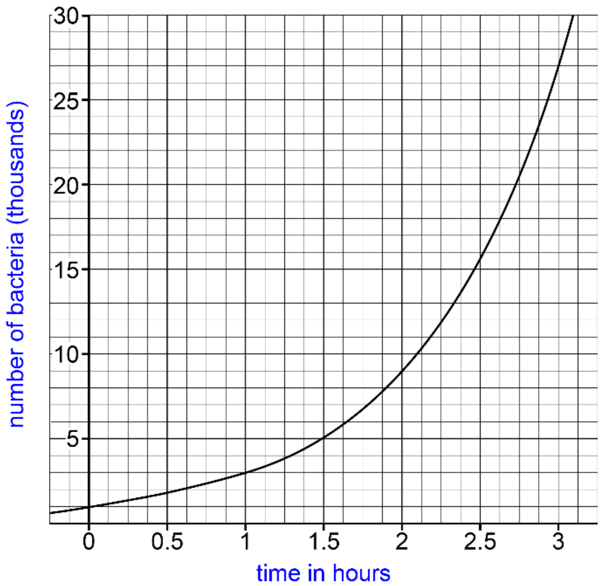
Now check that V = $\frac{2.5}{P}$ or (rearranging) PV ≈ 2.5 for 3 or 4 points: P

P	V	PV (to 1 d.p.)
1.0	2.50	2.5
2.5	1.00	2.5
5.0	0.50	2.5
6.0	0.42	2.5

Exponential graphs

An experiment measures the growth of bacteria in a colony.

The relationship between the number, n, of thousands of bacteria in the colony and time in hours, t, is of the form $n = k^t$. The graph shows the relationship between n and t from the start of the experiment.



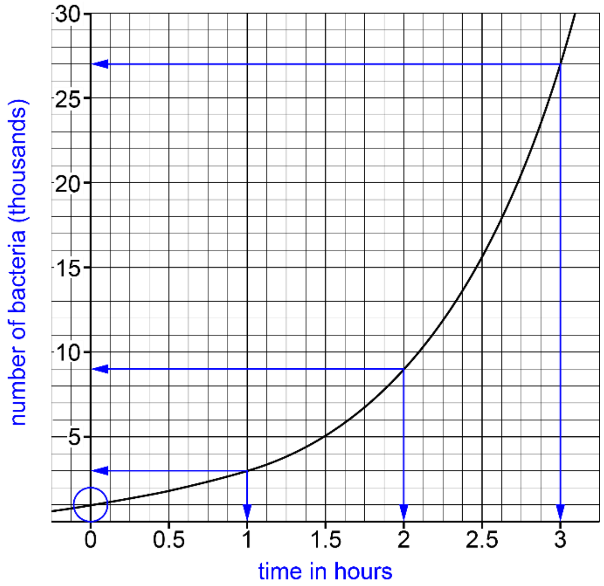
How many bacteria are in the colony at the start of the experiment?

Fill in the gap in this sentence: The number of bacteria in the colony is multiplied by a factor of __ every hour.

Write an equation connecting n and t.

How many bacteria would be in the colony at the end of the 4th hour?

At the start of the experiment, t = 0, so read the value of n when t = 0. This is 1, so 1000 bacteria.
After 1 hour there are 3000 bacteria, after 2 hours there are 9000 bacteria, and after 3 hours 27 000, so the population is multiplied by 3 every hour.

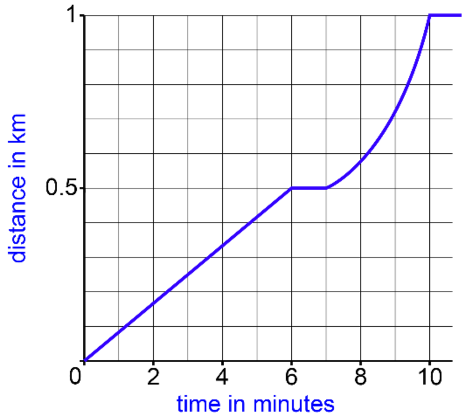


The equation is $n = 3^t$ as the multiplying factor is 3 every hour.

At the end of 4 hours there will be 3 × 27 000 = 81 000 bacteria.

Non-standard graphs

The distance-time graph shows Ania's journey from home to school. The vertical axis shows her distance from home in km and the horizontal axis shows the time since she leaves home.



School is 1 km from her house.

She walks, without stopping, from home to the shop, where she waits for her friend Sara. They leave the shop as soon as Sara arrives, but realise that they are late for school, so they run from the shop to school.

How far is the shop from Ania's house in metres?

For how long does Ania wait for Sara at the shop?

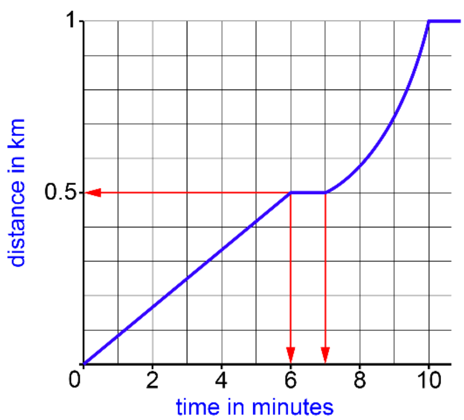
Why is Ania's journey from home to the shop shown as a straight line?

What is Ania's walking speed from home to the shop, in metres per minute?

Why is the journey from the shop to school shown as a curve?

What is Ania's average speed for the journey from the shop to school in metres per minute?

When Ania reaches the shop, she stops walking, so the graph is flat because the distance from home is not increasing. This happens at minute 6, and at 0.5 km from home (500 m).



While she waits, her distance from home is not increasing and the graph is flat. This happens between minute 6and minute 7, so for 1 minute.

The straight line shows that the distance from home is increasing at a constant rate, so she is walking at constant speed.

Ania walks the 0.5 km to the shop in 6 minutes. 0.5 km is 500 m. If she walks 500 m in 6 minutes, the speed is $\frac{500}{6}$ m/min= $83\frac{1}{3}$ m/min

The section of the graph between 7 and 10 minutes is curved, which means that Ania is moving, but not at constant speed. The increasing slope of the curve means that she is accelerating.

The distance from the shop to school is 500m. She takes 3 minutes to do this so her average speed is $\frac{500}{3}$ = $166\frac{2}{3}$ m/min

M4.14

Calculate or estimate gradients of graphs and areas under graphs (including quadratic and other non-linear graphs), and interpret results in cases such as distance–time graphs, speed–time graphs and graphs in financial contexts.

Gradient of straight-line graphs

The gradient of the straight line $y = mx + c$ is m.

If (a,b) and (c,d) are two points on a straight line, the gradient of the line is $\frac{d-b}{c-a}$

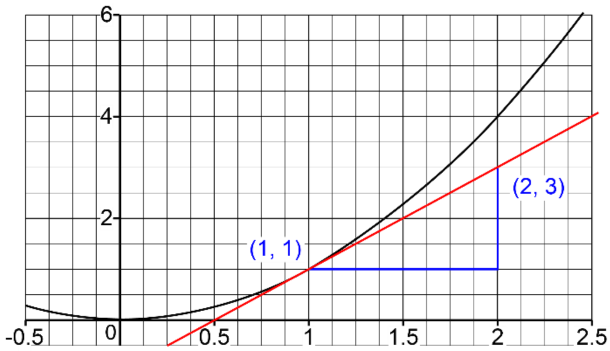
Gradient of curves

The gradient of a curve at a point is equal to the gradient of the tangent to the curve at that point. The tangent to the curve at a point is a line which touches the curve at that point and has the same gradient (slope) as the curve at that point.

The graph shows the curve $y = x^2$ and the tangent to the curve at the point $(1, 1)$. That tangent also passes through the point $(2, 3)$.

The gradient of the tangent is $\frac{3-1}{2-1} = 2$

Therefore, the gradient of the curve at the point $(1, 1)$ is 2.



Note: The phrase ‘area under a graph’ usually refers to the area between the graph and the horizontal axis.

Area under a straight-line graph

The area under a straight-line graph will be made up of some combination of triangles, rectangles and trapezia, and can be found by summing the formulae for the area of these shapes.

Approximate areas under curves

To approximate the area between a curve and an axis, divide the area into strips perpendicular to the axis, the strips are usually trapezia or triangles and should be chosen to give the best fit to the area. At this stage, strips do not have to have the same width.

Interpretation of gradient

If a measure such as distance is plotted on the vertical axis and another measure, such as time, is plotted on the horizontal axis, the gradient of the graph will give the measure:

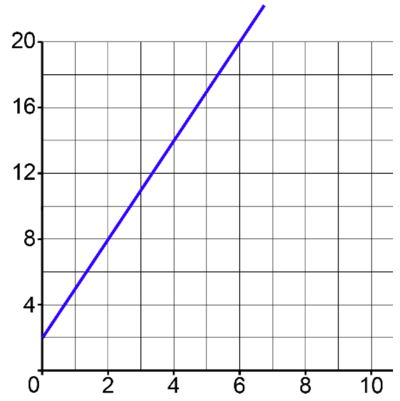
$$\frac{\text{change in distance}}{\text{change in time}} = \text{speed}$$

Interpretation of area under a curve

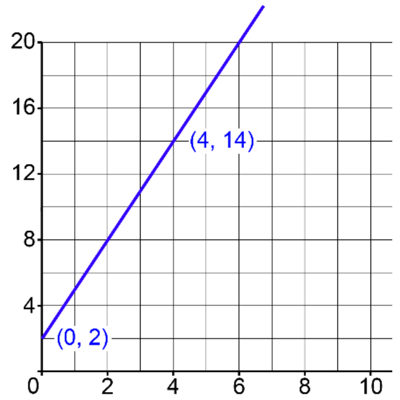
If a measure such as speed is plotted on the vertical axis, and another measure, such as time, is plotted on the horizontal axis, the area under the graph will give the measure: speed × time = distance

Gradient of straight-line graphs

Find the gradient of the line segment shown:



Pick two points on the line, preferably points which have integer coordinates, and give their coordinates.



The coordinates are $(0, 2)$ and $(4, 14)$. The gradient is $\frac{14-2}{4-0} = \frac{12}{4} = 3$

Gradient of curves

Find the gradient of the curve $y = x^2$ at the point $(2, 4)$.

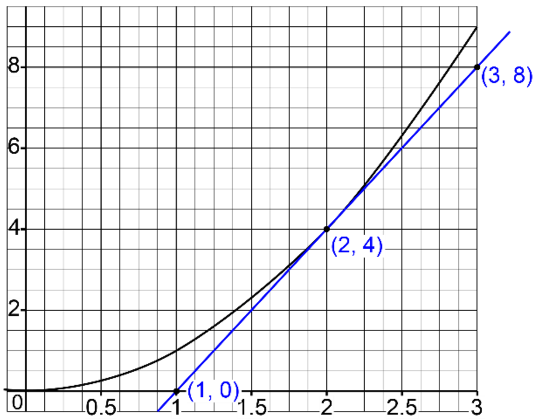
Draw the curve $y = x^2$ using the largest scale possible for $0 \leq x \leq 3$ plotting several points around the given point:

0	1	1.5	1.8	1.9	2	2.1	2.2	2.5	3
0	1	2.25	3.24	3.61	4	4.41	4.84	6.25	9

Draw a tangent at the point $(2, 4)$.

Note: The best way to do this is to place your ruler across the curve so that an equal amount of curve is showing either side of the point $(2, 4)$.Now slide the ruler towards the point $(2, 4)$, keeping equal amounts of curve either side of that point until you reach the point.

Pick two points on the tangent, preferably points with integer coordinates.



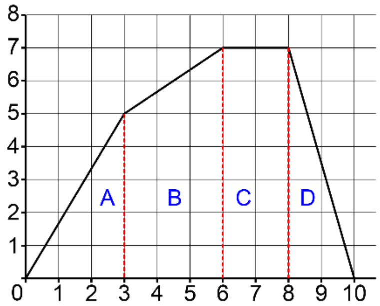
On the diagram, $(1, 0)$ and $(3, 8)$ are chosen.

Now find the gradient: $\frac{8-0}{3-1} = \frac{8}{2} = 4$

Area under a straight-line graph

Find the area under the graph shown.

Split the area into sections that you can easily find the area of:



The area of triangle A is $\frac{3\times5}{2}$ = 7.5

The area of trapezium B is $\frac{3}{2}\,(5 + 7) = 18$

The area of rectangle C is 2 × 7 = 14

The area of triangle D is $\frac{2\times7}{2} = 7$

Total area is 7.5 + 18 + 14 + 7 = 46.5 squares

Approximate areas under curves

Find the approximate area under the curve shown for $0 \leq x \leq 2.5$ Give your answer correct to 1 d.p.

Divide the curve, using vertical strips with straight line boundaries, into areas which approximate as closely as possible to the curve.

In this case, 2 trapezia and a triangle are used.

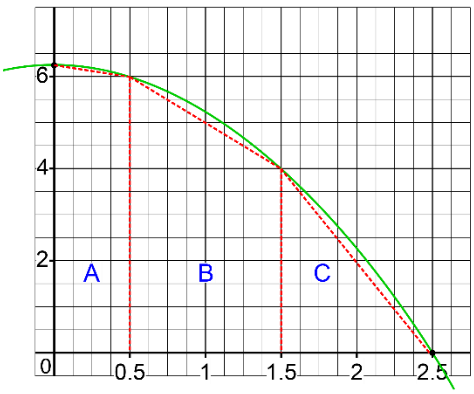
The area of trapezium A is $\frac{0.5}{2}$(6.25 + 6) = 3.0625

The area of trapezium B is $\frac{1}{2}$(6 + 4) = 5

The area of triangle C is $\frac{1 \times 4}{2} = 2$

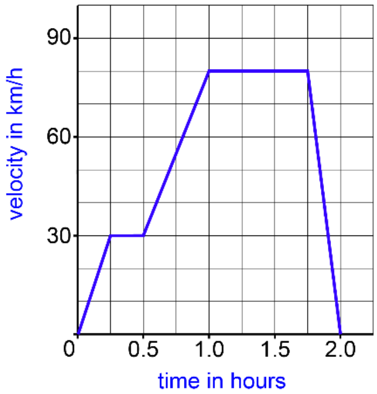
Total approximate area is A + B + C = 10.0625 which is 10.1 squares correct to 1 d.p.

(The actual area is 10.41 so the error is 3%)



Interpretation of area under a graph

The graph shows a speed–time graph of a car journey from Jai's home to his grandfather's home. The car is driven by the shortest route and the only stop that the car makes is at the end of the journey.



What is the distance, in km, of the shortest route from Jai's home to his grandfather's house?

The area under the graph is speed × time which is distance in km. Split the area into easy-to-calculate pieces.

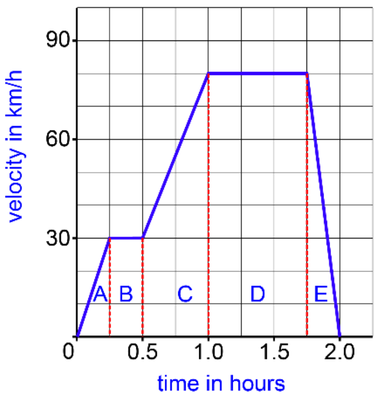
Area of triangle A is $\frac{0.25\times 30}{2} = 3.75$

Area of rectangle B is 0.25 × 30 = 7.5

Area of trapezium C is $\frac{0.5}{2}$(30 + 80) = 27.5

Area of rectangle D is 0.75 × 80 = 60

Area of triangle E is $\frac{0.25\times 80}{2} = 10$



Total distance is A + B + C + D + E = 3.75 + 7.5 + 27.5 + 60 + 10 = 108.75 km

M4.15

Set up and solve, both algebraically and graphically, simple equations including simultaneous equations involving two unknowns; this may include one linear and one quadratic equation.

Solve two simultaneous equations in two variables (linear/linear or linear/quadratic) algebraically.

Find approximate solutions using a graph.

Translate simple situations or procedures into algebraic expressions or formulae; for example, derive an equation (or two simultaneous equations), solve the equation(s) and interpret the solution.

In solving an equation, the aim is to isolate the unknown quantity and find its value.

This can be achieved by:

Adding, subtracting, multiplying or dividing both sides of the equation by the same quantity and then simplifying.

Solutions of simultaneous equations are answers which satisfy both the equations.

Setting up equations

To set up an equation, you need to identify the unknown quantity and then write a mathematical statement involving the unknown.

Graphical solution of simultaneous linear equations

To solve simultaneous equations graphically, graph the equations and find the intersection point(s).

Algebraic solution of simultaneous linear equations

To solve linear simultaneous equations in two unknowns, it is necessary to eliminate one unknown. The two main methods of solution are substitution or using a linear combination of the two equations, but there are other methods.

Algebraic solution of one linear and one quadratic equation

To solve one linear and one quadratic equation, use the method of substitution. Use the linear equation to find an expression for one of the variables in terms of the other variable, then substitute this expression into the quadratic equation and solve the quadratic. Generally, there will be two solutions for each of the variables. The solutions should be presented in pairs.

Graphical solution of one linear and one quadratic equation

An approximate solution of one linear and one quadratic equation can be found by graphing the two equations and finding the point or points of intersections. The solutions should be presented in pairs.

Modelling real situations

It is possible to model real situations using equations and interpret the solutions.

Setting up equations

In a shop, the cost of one pen is 4 times the change from £15 after buying one pen. What is the cost of one pen?

The unknown is the cost of a pen. Let this be £x.

The change from £15 after buying a pen is £$(15 - x)$. This means the equation is: $x = 4(15 - x)$ and this equation must be solved for x.

$x = 60 - 4x$ Multiply out the bracket $x + 4x = 60 - 4x + 4x$ Add 4x to both sides of the equation

$5x = 60$ Simplify

$5x \div 5 = 60 \div 5$ Divide both sides of the equation by 5

$x = 12$ Simplify

Graphical solution of simultaneous linear equations

Solve the following pair of simultaneous equations graphically:

$$3x + 2y = 9 \qquad x - y = -2$$

The solution is the point of intersection of the lines

$3x + 2y = 9\ x - y = -2$

Find 2 points on the line $3x + 2y = 9$

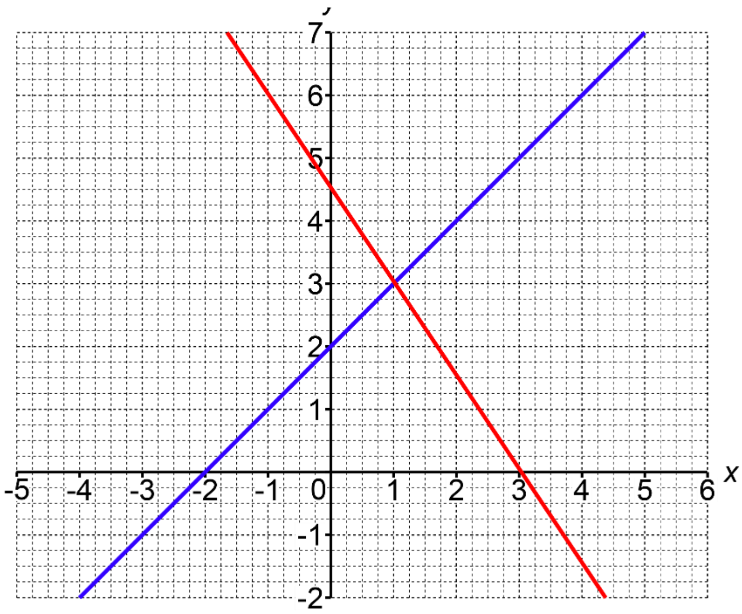
When $y = 0, x = 3$ and when $x = 0, y = 4.5$

Find 2 points on the line $x - y = -2$

When $x = 0, y = 2$ and when $y = 0, x = -2$

Plot both these lines and read off the x and y values of the point of intersection.

$x = 1$ and $y = 3$



Algebraic solution of simultaneous linear equations

Solve the simultaneous equations:

$$3x - 4y = 7 \quad x + 2y = 9$$

$3x - 4y = 7$ (i)
$x + 2y = 9$ (ii)
$x = 9 - 2y$ (iii) Rearrange (ii) to get x in terms of y
$3(9 - 2y) - 4y = 7$ Substitute (iii) into (i)
$27 - 6y - 4y = 7$ Multiply out brackets
$27 - 10y = 7$ Simplify
$27 - 10y - 27 = 7 - 27$ Subtract 27 from both sides
$-10y = -20$ Simplify
$-10y \div -10 = -20 \div -10$ Divide both sides by -10
$y = 2$
$x = 9 - 2y = 9 - 4 = 5$ Substitute $y = 2$ into (iii) to find x

Check by putting your answers into equation (i) giving $3 \times 5 - 4 \times 2 = 7$ which is correct.

Algebraic solution of one linear and one quadratic equation

Solve the simultaneous equations:

$$x + 4y = 7 \qquad x^2 + 4xy + 2y^2 = 1$$

$x + 4y = 7$ (i)

$x^2 + 4xy + 2y^2 = 1$ (ii)

$x + 4y = 7$

$x + 4y - 4y = 7 - 4y$ Subtract $4y$ from both sides of equation (i) to express x in terms of y

$x = 7 - 4y$ (iii)

$(7 - 4y)^2 + 4(7 - 4y)y + 2y^2 = 1$ Substitute (iii) into (ii) to eliminate x

$49 - 56y + 16y^2 + 28y - 16y^2 + 2y^2 = 1$ Multiply out the brackets

$2y^2 - 28y + 48 = 0$ Simplify by collecting like terms

$y^2 - 14y + 24 = 0$ Divide every term by 2

$(y - 12)(y - 2) = 0$ Factorise the quadratic

$y = 12$ or $y = 2$

If $y = 2$ then $x = 7 - 4y = -1$

If $y = 12$ then $x = 7 - 4y = -41$

The solutions are:

$x = -1, y = 2$ and $x = -41, y = 12$

Remember to check your solutions by substituting into the original equations.

Graphical solution of one linear equation and one quadratic equation

Using a graphical method, find all approximate solutions of the simultaneous equations correct to 1 decimal place in the range $-5 \leq x \leq 3$:

$$x - y = -2 \qquad y = x2 + 4x + 1$$

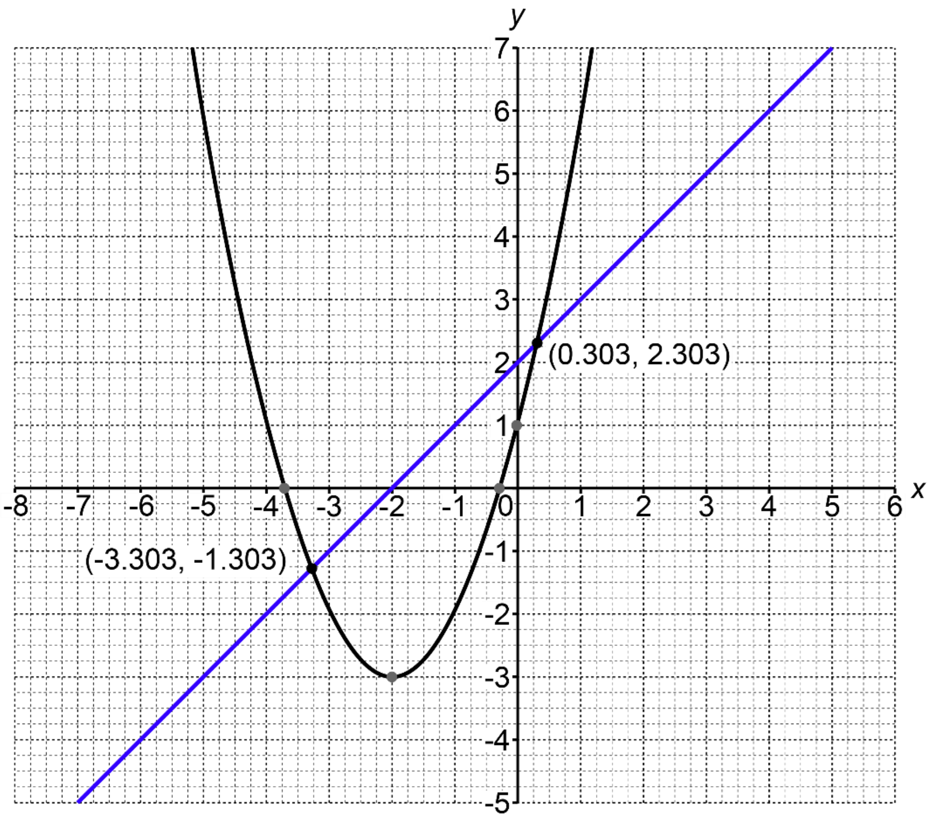
Plot both the graphs in the range stated:

For $x - y = -2$, when $x = 0, y = 2$ and when $y = 0, x = -2$

For $y = x^2 + 4x + 1$

x	-5	-4	-3	-2	-1	0	1	2	3
x^2	25	16	9	4	1	0	1	4	9
$4x$	-20	-16	-12	-8	-4	0	4	8	12
1	1	1	1	1	1	1	1	1	1
y	6	1	-2	-3	-2	1	6	13	22

Draw the graphs and read off the points of intersection. The accurate answers are shown.



The approximate solutions are $x = -3.3, y = -1.3$ or $x = 0.3, y = 2.3$

Modelling real situations

Two shades of purple paint are made up entirely of red paint and blue paint.

10 litres of imperial purple paint is made up of 7 litres of blue paint and 3 litres of red paint and costs £16.50.

10 litres of royal purple paint is made up of equal quantities of red and blue paint and costs £17.50.

How much would 10 litres of red paint cost?

Let £x be the cost of 1 litre of blue paint and £y be the cost of 1 litre of red paint.
For imperial purple: $7x + 3y = 16.50$ (i)
For royal purple: $5x + 5y = 17.50$ (ii)
Multiply every term in (ii) by $\frac{3}{5}$ $3x + 3y = 10.50$ (iii)
(i) – (ii) $4x + 0 = 6$
Divide both terms by 4: $x = 6 \div 4 = 1.5$
Substitute in (i) $10.5 + 3y = 16.5$
Subtract 10.5 $10.5 + 3y - 10.5 = 16.5 - 10.5$
Simplify $3y = 6$ so $y = 2$
10 litres of red paint costs 10 × £2 = £20
Remember to check that the solution works in both equations.

M4.16

Solve quadratic equations (including those that require rearrangement) algebraically by factorising, by completing the square, and by using the quadratic formula.

Know the quadratic formula:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Find approximate solutions of quadratic equations using a graph.

Solve quadratic equations by factorising

Expressing a quadratic in the form $(ax + b)(cx + d) = 0$, where a, b, c and d are real numbers, and solving $ax + b = 0$ and $cx + d = 0$

Completing the square

$$x^2 + ax = \left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4}$$

Solve quadratic equations by completing the square

Expressing a quadratic in the form $(ax + b)^2 = c$, where a, b and c are real numbers, and then solving

Solve quadratic equations by using the quadratic formula

Expressing a quadratic in the form $ax^2 + bx + c = 0$ and then solving for x using the formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Note: You need to be able to recall this formula.

Finding approximate solutions of quadratic equations using a graph

Approximate solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found by drawing a graph of the quadratic function $y = ax^2 + bx + c$ and finding the values of x for which $y = 0$

Solve quadratic equations by factorising

Solve the equation: $6x^2 - 7x - 3 = 0$

Factorise the quadratic expression by whichever method you prefer.

Split the middle term by finding two numbers which multiply to $6 \times -3 = -18$ and add up to -7. The numbers are -9 and $+2$.

$$6x^2 - 7x - 3 = 6x2 - 9x + 2x - 3 = 3x(2x - 3) + 1(2x - 3) = (2x - 3)(3x + 1)$$

If $(2x - 3)(3x + 1) = 0$ then either $2x - 3 = 0$ or $3x + 1 = 0$

If $2x - 3 = 0$ then $x = +\frac{3}{2}$

If $3x + 1 = 0$ then $x = -\frac{1}{3}$

Note: A quadratic equation may have no real solutions, one real solution, or two real solutions.

Solve quadratic equations by factorising – disguised quadratics

Solve the equation: $\frac{3}{x^2} + \frac{7}{x} = 6$

Multiply every term in the equation by x^2:

$3 + 7x = 6x^2$

Rearrange the equation: $6x^2 - 7x - 3 = 0$

This is now the quadratic equation that we solved in the previous example.

Solve quadratic equations by factorising – disguised quadratics

Solve the equation: $6p^6 = 7p^3 + 3$ to find expressions for p

The equation has 3 terms and $p^6 = (p^3)^2$

Rearrange: $6(p^3)^2 - 7p^3 - 3 = 0$

This can be written as $6x^2 - 7x - 3 = 0$ where $x = p^3$

Solving as in the first example:

$x = \frac{3}{2}$ or $x = -\frac{1}{3}$

so $p^3 = \frac{3}{2}$ or $p^3 = -\frac{1}{3}$

so $p = \sqrt[3]{\frac{3}{2}}$ or $p = \sqrt[3]{-\frac{1}{3}}$

Completing the square

Express $x^2 - 3x$ as a difference of two squares in the form $(x + a)^2 - a^2$ by completing the square.

$$(x + a)^2 - a^2 = x^2 + 2ax + a^2 - a^2 = x^2 + 2ax$$

Comparing $x^2 + 2ax$ with $x^2 - 3x$:

$2a = -3$ so $a = -\frac{3}{2}$

$\left(x - \frac{3}{2}\right)^2 = x^2 - 3x + \frac{9}{4} = x^2 - 3x + \left(\frac{3}{2}\right)^2$

$x^2 - 3x = \left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$

Solve quadratic equations by completing the square

Solve the equation: $x^2 - 4x - 5 = 0$ by completing the square.

You need to complete the square for $x^2 - 4x$.
$x^2 - 4x = (x - 2)^2 - 4$
If $x^2 - 4x - 5 = 0$ then $(x - 2)^2 - 4 - 5 = 0$
$(x - 2)^2 - 9 = 0$ so $(x - 2)^2 = 9$
and (taking the square root of the whole equation) $x - 2 = \pm 3$
If $x - 2 = +3$ then $x = 5$
If $x - 2 = -3$ then $x = -1$
So $x = 5$ or $x = -1$

Solve quadratic equations by using the quadratic formula

Solve the quadratic equation $3x^2 - 4x = 5$ using the quadratic formula. Leave your answer in surd form.

Write the equation in the form $ax^2 + bx + c = 0$:
$3x^2 - 4x - 5 = 0$ so $a = 3, b = -4$ and $c = -5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{(-(-4) \pm \sqrt{((-4)^2 - 4 \times 3 \times -5))}}{2 \times 3}$$

$$= \frac{4 \pm \sqrt{76}}{6} = \frac{4 \pm 2\sqrt{19}}{6} = \frac{4 \pm 2\sqrt{19}}{6}$$

So $x = \frac{2 \pm \sqrt{19}}{3}$

M4.17

Solve linear inequalities in one or two variables.

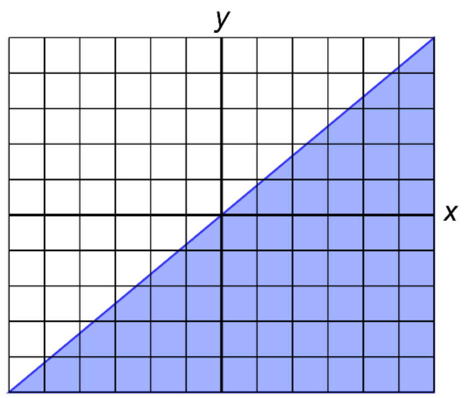
Represent the solution set on a number line, or on a graph, or in words.

Symbols and labelling conventions

Single variable

$<$	Less than (looks like an L for less)	$x < 4$ defines points on a number line such that x can take any value less than but not including 4. It is shown on a number line as:  The open circle at 4 means that 4 is not included in the solution set.
$>$	Greater than	$x > 4$ defines points on a number line such that x can take any value greater than but not including 4. It is shown on a number line as:  The open circle at 4 means that 4 is not included in the solution set.
$\leq$	Less than or equal to	$x \leq 4$ defines points on a number line such that x can take any value less than 4 and the value 4. It is shown on a number line as:  The solid circle at 4 means that 4 is included in the solution set.
$\geq$	Greater than or equal to	$x \geq 4$ defines points on a number line such that x can take any value greater than 4 and the value 4. It is shown on a number line as:  The solid circle at 4 means that 4 is included in the solution set.

Two variables



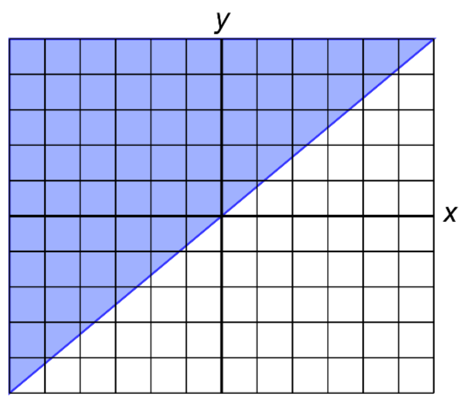
The shaded area of the graph shows the region $x < y$

The boundary is the line $x = y$

The dotted line at the edge of the shaded region shows that points on the line $x = y$ are not included.

If the diagram had a solid line for $x = y$ the diagram would show the inequality $x \leq y$

When solving problems graphically involving more than one inequality, it is often clearer to shade out the unwanted region of the graph and leave the required region unshaded. Either way is correct, as long as you state clearly what you are doing.



The shaded area of the graph shows the region $x \geq y$

The solid line at the edge of the shaded region shows that the points on the line $x = y$ are to be included.

Simplifying inequalities

If $x < y$

Adding or subtracting a real number from both sides of the inequality leaves the inequality sign unchanged.	$x - 4 < y - 4$
Multiplying or dividing both sides of the inequality by a positive real number leaves the inequality sign unchanged.	$\frac{3}{4}x < \frac{3}{4}y$
When x and y are both positive or both negative, inverting both sides of the inequality changes the direction of the inequality sign.	$\frac{1}{x} > \frac{1}{y}$
Multiplying or dividing both sides of the inequality by a negative real number changes the direction of the inequality sign.	$-\frac{3}{4}x > -\frac{3}{4}y$

Combining inequalities

Combined inequalities are of the form

$a < b < c$ or $a > b > c$
or
$a \leq b \leq c$ or $a \geq b \geq c$

The important thing is that both inequality signs are 'pointing' the same way. $a < b > c$ is not correct.

The inequalities $x > -7$ and $x \leq 10$ define a single continuous range which can be written as $-7 < x \leq 10$.

The inequalities $x < -7$ and $x \leq 10$ define a single continuous range which can be written as $x < -7$

The inequalities $x < -7$ and $x \geq 10$ do not define a single continuous range, and cannot be written as a single combined inequality.

The inequalities $x > -7$ and $x < -10$ do not define a single continuous range, and cannot be written as a single combined inequality.

The inequality $-7 < x < -10$ is incorrect as it states that $-7 < -10$

Inequalities in 2 variables

Inequalities in 2 variables are usually solved graphically.

Simple single linear inequality

Solve the inequality $3x - 4 > 8$ and show the result on a number line.

$3x - 4 > 8$

$3x - 4 + 4 > 8 + 4$ Add 4 to both sides

$3x > 12$ Simplify

$3x \div 3 > 12 \div 3$ Divide both sides by 3

$x > 4$

Draw the number line with an unfilled circle at 4 to show that it is not included.

Two sided linear inequality

Solve the inequality $20 > 3x - 4 > 8$ and show the result on a number line.

$20 \geq 3x - 4 > 8$

$20 + 4 \geq 3x - 4 + 4 > 8 + 4$ Add 4 to all three sections

$24 \geq 3x > 12$ Simplify

$24 \div 3 \geq 3x \div 3 > 12 \div 3$ Divide all three sections by 3

$8 \geq x > 4$

Draw the number line with an unfilled circle at 4 to show that it is not included and a filled circle at 8 to show that it is included. Draw the number line with the numbers increasing from left to right as usual even though the 8 comes before the 4 in the answer.

Finding a range of values from two inequalities

Find the range of values of x for which both these inequalities are valid:

$3x - 4 < 8$ and $3x - 2 > -8$

Solve the two inequalities separately:

$3x - 4 < 8$

$3x - 4 + 4 < 8 + 4$

$3x < 12$

$3x \div 3 < 12 \div 3$

$x < 4$ (i)

$3x - 2 > -8$

$3x - 2 + 2 > -8 + 2$

$3x > -6$

$x > -2$ (ii) (i) and (ii) can be written as the single inequality $-2 < x < 4$ because they can be represented as a single line on a number line:

Finding a range of values from two inequalities

Find the range of values of x for which both these inequalities are valid:

$3x - 4 < 8$ and $3x - 2 < -8$

Solve the two inequalities separately:

$3x - 4 < 8$

$3x - 4 + 4 < 8 + 4$

$3x < 12$

$3x \div 3 < 12 \div 3$

$x < 4$ (i)

$3x - 2 < -8$

$3x - 2 + 2 < -8 + 2$

$3x < -6$

$x < -2$ (ii)

(i) and (ii) can be combined to give the single inequality $x < -2$ which is the range valid for both inequalities.

Finding a range of values from two inequalities

Find, if possible, the range of values of x for which both these inequalities are valid:

$3x + 2 < -10$ and $3x - 4 > 8$

Solve the two inequalities separately:

$3x + 2 < -10$

$3x + 2 - 2 < -10 - 2$

$3x < -12$

$3x \div 3 < -12 \div 3$

$x < -4$ (i)

$3x - 4 > 8$

$3x - 4 + 4 > 8 + 4$

$3x > 12$

$x > 4$ (ii)

There is no range of values for which both inequalities are valid.

More complicated inequalities in a single variable

Solve the inequality $3x - 4 < 2(2 + 3x)$

$3x - 5 < 4 + 6x$ Multiply out the bracket

$3x - 5 - 6x < 4 + 6x - 6x$ Subtract $6x$ from both sides of the inequality

$-3x - 5 + 5 < 4 + 5$ Simplify and add 5 to both sides

$$-3x < 9$$

$3x > -9$ Multiply both sides by -1 reversing the inequality sign

$x > -3$ Divide both sides by 3

Inequalities in 2 variables

Shade the region for which $3x + 4y \leq 12$

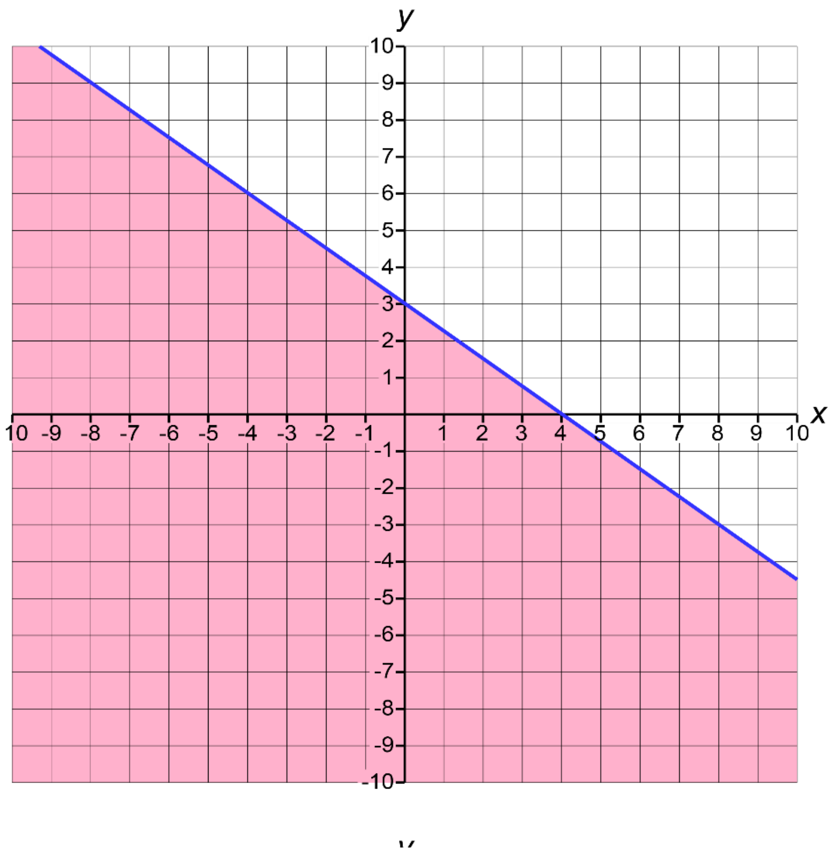
The boundary between $3x + 4y < 12$ and $3x + 4y > 12$ is the line $3x + 4y = 12$ so draw this line.
Check points above and below the line.

At $(0, 0)$, $3x + 4y < 12$ as $0 < 12$

Check a point above the line, at $(0, 6)$, $3x + 4y > 12$ as $24 > 12$

The region required is the one below the line.

As the inequality is $3x + 4y \leq 12$ then the boundary line is included and should be shown as a solid line.



Solving simultaneous inequalities graphically

Draw the region for which all the following inequalities are satisfied:

$3x + 4y \leq 12$
$y \geq x + 6$
$x > -7$

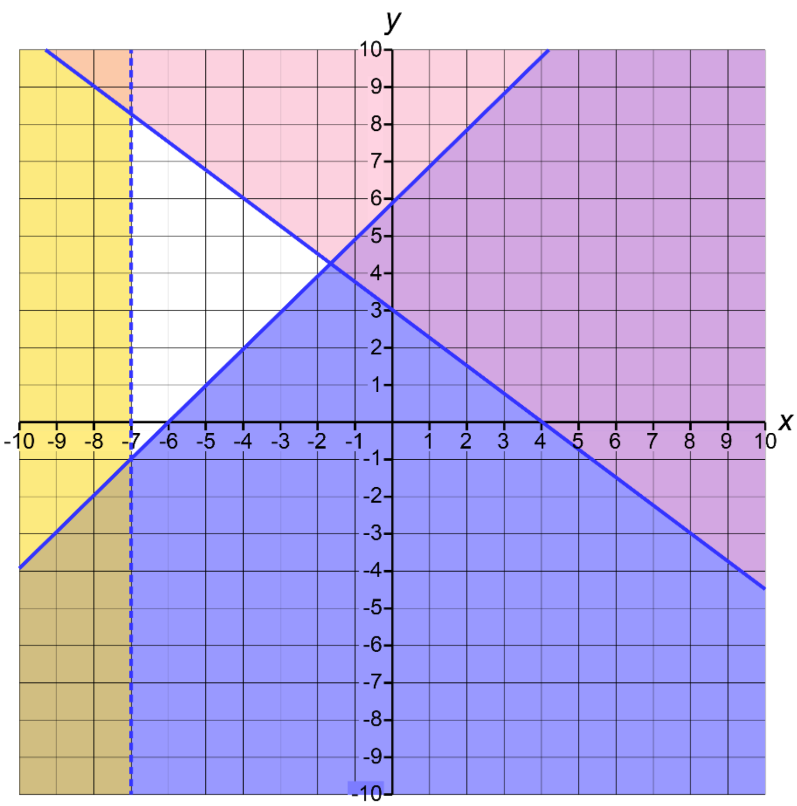
With more than one inequality, it is usually easier to shade out the unwanted regions so you can see the required region clearly.

$3x + 4y \leq 12$
Draw the line $3x + 4y = 12$ and shade out the 'greater than' region which is the region above the line. The line is solid as ≤ means that points on the line are included.

$y \geq x + 6$
Draw the line $y = x + 6$ and shade out the region below the line which is the 'less than' region. The line is solid as ≥ means that points on the line are included.

$x > -7$
Draw the line $x = -7$ and shade out the region to the left of the line. The line is broken as $>$ means that points on the line are not included.

The solution is the unshaded region with the solid lines on the boundary of the unshaded region.



M4.18

Generate terms of a sequence using term-to-term or position-to-term rules.

A sequence is a list of terms together with a rule for generating them.

Sequences can be generated using a term-to-term rule or position-to-term rule.

A term-to-term rule indicates how to move from one term in the sequence to the next term in the sequence.

A position-to-term rule indicates how to move from the position in the sequence to the term in the sequence. For example, how to find the term in position 20 of the sequence.

Examples of a term-to-term rule

Sequences may be described by giving the first term and the term-to-term rule. For example, the sequence with first term 3 and term-to-term rule +4 is 3, 7, 11, 15, 19, … as you are adding 4 each time to move to the next term in the sequence.
Term-to-term rules may also be written using the notation t_1 for the first term in the sequence and t_n for the nth term in the sequence. For example, $t_1 = 7$ and $t_n+1 = t_n–2$ generates the sequence 7, 5, 3, 1, -1, …

You should be able to:

- generate sequences using a term-to-term rule find the next term in a sequence.

Example of a position-to-term rule

2n–1 is an example of a position-to-term rule.
For this sequence the 3rd term is 5 since 2 × 3 − 1 = 5. This sequence has an 8th term of 15 since 2 × 8 − 1 = 15

You should be able to:

- generate sequences using a position-to-term rule find the position-to-term rule for a sequence.

Using a term-to-term rule to generate a sequence

Find the next 4 terms in this sequence: $t_1 = 3$; $t_n+1 = 2t_n–1$

$t_1 = 3$ $t_2 = (2 × 3) – 1 = 5$ $t_3 = (2 × 5) – 1 = 9$ $t_4 = (2 × 9) – 1 = 17$ $t_5 = (2 × 17) – 1 = 33$

Deciding whether a number is in a particular sequence

Is 272 in the sequence 2, 6, 10, 14, …? Explain your answer.

The first term is 2 and term-to-term rule is +4.
If a term is in the sequence, we should be able to subtract 2 and obtain a multiple of 4. 272 is not a term in the sequence as 272 – 2 = 270, which is not a multiple of 4.

Finding terms in a sequence from the n**th term rule**

Find the 4th and 10th terms for the sequence whose nth term rule is 2n + 3

n = 4
4th term = (2 × 4) + 3 = 11 n = 10
10th term = (2 × 10) + 3 = 23

Finding the position of a particular term in a sequence
In the sequence $n^2 + 2n$, which term is 35?

Solve the equation $n^2 + 2n = 35$
$n^2 + 2n - 35 = 0$
$(n + 7)(n - 5) = 0$
n = –7 or 5, but since n must be positive, n = 5 and 35 is the 5th term.

M4.19

Deduce expressions to calculate the n^{th} term of linear or quadratic sequences.

If we know a list of terms in a sequence, we can find the nth term which is also the position-to-term rule for working out any term in the sequence.

For a linear sequence, the terms increase by the same amount each time. We say there is a constant difference between the terms.

For a quadratic sequence, the terms are generated using an nth term which is in the form $an^2 + bn + c$ where either b or c could be 0.

The nth term for the linear sequence 1, 3, 5, 7, … is 2n–1

The nth term for the quadratic sequence 1, 7, 17, 31, … is $2n^2–1$

Finding an nth term for a linear sequence

Find the nth term for the sequence 2, 5, 8, 11, …

n	1		2		3		4
term	2		5		8		11
difference		+3		+3		+3	

A constant difference of +3 between terms means that the nth term includes 3n Each term is 1 less than 3 × n so the nth term is 3n–1

Finding the nth term for a decreasing linear sequence

Find the nth term for the sequence 14, 8, 2, –4, …

We write out the position numbers (n) and the terms in the sequence but leave a blank row in between.

n	1	2	3	4
term	14	8	2	-4

As the difference between terms is –6, we know the nth term involves –6n and we can fill in the middle row with values for –6n:

n	1	2	3	4
-6n	-6	-12	-18	-24
term	14	8	2	-4

To get from each of the values in the middle row to the terms in the 3rd row we add 20, so the nth term is –6n + 20

Finding the nth term for a linear sequence with fractional coefficient

Find the nth term for the sequence $5\frac{1}{2}$, 6, $6\frac{1}{2}$, 7, …

n	1	2	3	4
term	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7
difference		$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$

As the difference between terms is $+\frac{1}{2}$, the nth term will include $\frac{1}{2}$ n. Each term is 5 more than $\frac{1}{2}$n so the nth term is $\frac{1}{2}$n + 5

Finding the nth term for a quadratic sequence

Find the nth term for the sequence 2, 5, 10, 17, …

Method 1

We can work out the difference between terms, but this time we need an extra row in the table before we get a constant difference.

n	1		2		3		4		5		6
term	2		5		10		17		26		37
1st difference		+3		+5		+7		+9		+11	
2nd difference			+2		+2		+2		+2		+2

The second difference is constant, so the nth term is in the form $an^2 + bn + c$
When n = 1, a + b + c = 2 (the 1st term). When n = 2, 4a + 2b + c = 5 (the 2nd term).
The difference between these terms is 3a + b = 3
If we do the same for the 2nd and 3rd terms we get:
9a + 3b + c – (4a + 2b + c) = 10–5 so 5a + b = 5
Solving these two simultaneous equations gives a = 1, b = 0
Substituting back into a + b + c = 2 gives us c = 1 so the nth term for the sequence is $n^2 + 1$

Method 2

We know there is n^2 in the rule as the 2nd differences are constant whereas the 1st differences are not. To find the coefficient of n^2 we divide the second difference by two, so we can see that we have n^2 in the nth term. The sequence for n^2 would be 1, 4, 9, 16 so the terms are all 1 greater than n^2. This means that the nth term rule is $n^2 + 1$.

Finding the nth term for a quadratic sequence based on a multiple of n^2

Find the nth term for the sequence 1, 10, 25, 46, …

Method 1

n	1		2		3		4
term	1		10		25		46
1st difference		+9		+15		+21	
2nd difference			+6		+6		

As the 2nd difference is constant, we know this is a quadratic sequence and the nth term is in the form $an^2 + bn + c$

We can make some equations using different values of n:

When n = 1, a + b + c = 1 (the 1st term).

When n = 2, 4a + 2b + c = 10 (the 2nd term). The difference between these terms is 3a + b = 9

If we do the same for the 2nd and 3rd terms we get

9a + 3b + c–(4a + 2b + c) = 25–10 so 5a + b = 15

Solving these two simultaneous equations gives a = 3, b = 0

Substituting back into a + b + c = 1 gives c =–2 so the nth term is $3n^2–2$

Method 2

We know there is n^2 in the rule as the 2nd differences are constant whereas the 1st differences are not. To find the coefficient of n^2 we divide the second difference by two, so we can see that we have $3n^2$ in the nth term. The terms are all 2 smaller than $3n^2$ (which would be 3, 12, 27, 48, …) so the nth term is $3n^2–2$.

Finding the nth term for a quadratic sequence with terms in n^2 and n

Find the nth term for the sequence 5, 14, 27, 44, 65 , ...

Method 1

n	1		2		3		4		5
term	5		14		27		44		65
1st difference		+9		+13		+17		+21	
2nddifference			+4		+4		+4		

The 1st difference is not constant so we also need to calculate the 2nd difference. This is constant so the sequence is quadratic with nth term rule in the form $an^2 + bn + c$ We can make some equations using different values of n:

When n = 1, a + b + c = 5 (the 1st term).

When n = 2, 4a + 2b + c = 14 (the 2nd term)

When n = 3, 9a + 3b + c = 27 (the 3rd term)

Subtracting the first equation from the second one gives 3a + b = 9

Subtracting the second equation from the third one gives 5a + b = 13

Solving these simultaneously we get a = 2, b = 3, and substituting back into any of the equations we find that c = 0, so the nth term rule is $2n^2 + 3n$

Method 2

We know there is n^2 in the rule as the 2nd differences are constant whereas the 1st differences are not. To find the coefficient of n^2 we divide the second difference by two, so we can see that we have $2n^2$ in the nth term. We can compare the sequence that we have got to $2n^2$ to find out what needs to be added or subtracted.

term	5	14	27	44	65
$2n^2$	2	8	18	32	50
Add to $2n^2$	+3	+6	+9	+12	+15

We can then find the nth term for the amounts that have been added which is 3n.

Combining the $2n^2$ with the 3n we can see that the nth term is $2n^2 + 3n$.

M5. Geometry

M5.1

Use conventional terms and notation: points, lines, line segments, vertices, edges, planes, parallel lines, perpendicular lines, right angles, subtended angles, polygons, regular polygons and polygons with reflection and/or rotational symmetries.

Definitions

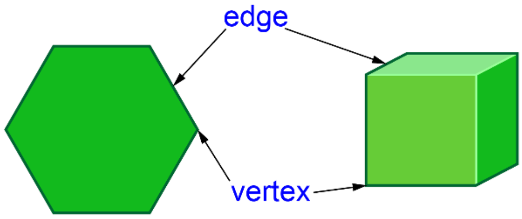
Point
A singular position. The position can be defined in various ways including by coordinates on a grid or by the intersection point of two lines.

Line
An infinitely long one-dimensional figure.

Line segment
A part of a line.

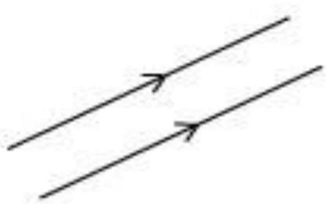
Vertex/vertices
A vertex is a corner. For flat shapes, it is where the edges meet. For cones, pyramids etc., all corners (including the point at the top) are called vertices.

Edge
The side of a polygon or polyhedron.



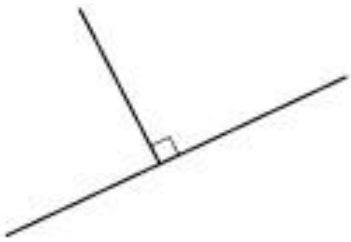
Plane
A flat surface.

Parallel lines
Lines which are always the same perpendicular distance apart. Parallel lines are indicated by arrows.



Perpendicular lines

Lines that are at right angles to each other. When the lines intersect, the right angle between the lines is generally shown.



Right angle

90°

Subtended angles

An angle subtended by an arc, line segment etc. is one whose two rays pass through the endpoints of the arc, line segment etc.

Polygon

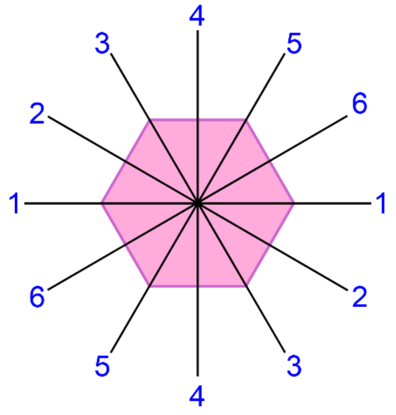
A closed plane shape with 3 or more straight sides.

Regular polygon

A polygon with all its sides equal and all its angles equal.

Symmetries of polygons

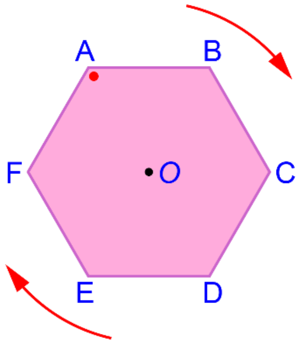
If you fold a two-dimensional shape along a line of reflection symmetry, one side of the shape will fold exactly onto the other side. If you place a mirror on the line of reflection symmetry of a two-dimensional shape, one side of the shape would be an exact reflection of the other side. A regular hexagon has 6 lines of symmetry.



If you draw around the outline of a shape and then rotate the shape about its centre, the order of rotational symmetry of the shape is the number of times the shape fits exactly into its outline when it is turned through 360°.

If the hexagon below is rotated through 360° about O, it will fit back onto its outline 6 times when the red dot is at vertices A, B, C, D, E and F.

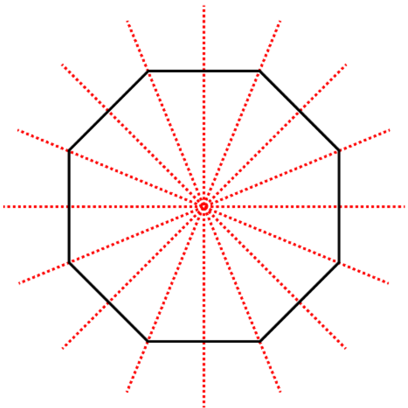
The hexagon has 6 lines of symmetry and order of rotational symmetry 6. Not all shapes have the same number of lines of symmetry as order of rotational symmetry, although regular polygons generally do.

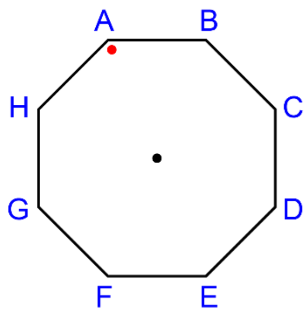


Symmetries of polygons

Describe the symmetries of a regular octagon.

The 8 lines of symmetry are shown, and the order of rotational symmetry is also 8.





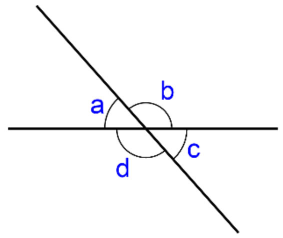
M5.2

Recall and use the properties of angles at a point, angles on a straight line, perpendicular lines and opposite angles at a vertex.

Understand and use the angle properties of parallel lines, intersecting lines, triangles and quadrilaterals.

Calculate and use the sum of the interior angles, and the sum of the exterior angles, of polygons.

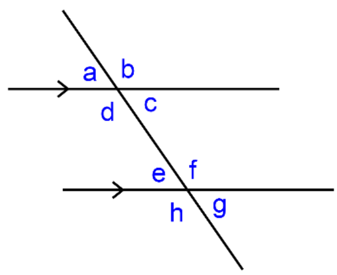
Properties of angles around a point, angles on a straight line, perpendicular lines and opposite angles at a vertex



The sum of the angles around a point is 360° (a + b + c + d = 360°)
The sum of the angles at a point on one side of a straight line is 180° (a + b = b + c = c + d = d + a = 180°)
Perpendicular lines are at right angles (90°) to each other.
Opposite angles at a vertex are equal (a = c and b = d)
You can use this to:

- find unknown angles at a point find unknown angles on a straight line find opposite angles at a vertex.

Angle properties of parallel lines



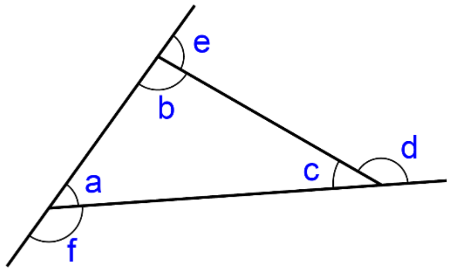
If a line crosses a pair of parallel lines then:

- alternate angles are equal (d = f and c = e)
- corresponding angles are equal (a = e, b = f, c = g and d = h) allied or co-interior angles are supplementary – add up to 180° (c + f = d + e = 180°).

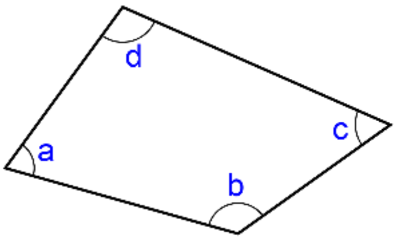
You can use this to:

- find unknown angles in parallel lines.

Angle properties of triangles and quadrilaterals



The sum of the angles of a triangle is 180° (a + b + c = 180°).
The exterior angle of a triangle is equal to the sum of the interior opposite angles (d = a + b and e = a + c and f = b + c).



The sum of the angles of a quadrilateral is 360° (a + b + c + d = 360°)

You can use this to:

- find unknown angles in a triangle

Interior and exterior angle sum of polygons

The sum, in degrees, of the interior angles of an n-sided polygon is 180 (n–2).

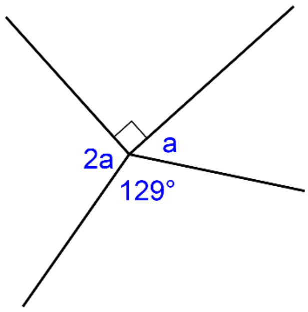
The sum of the exterior angles of any polygon is 360°

You can use this to:

- find unknown angles in polygons including exterior angles.

Finding unknown angles at a point

Find the value of a in the diagram.

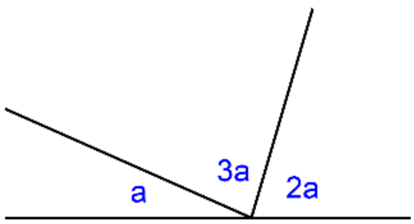


The sum of angles at a point = 360°

90° + a + 2a + 129° = 360° 3a = 141° a = 47°

Finding unknown angles on a straight line

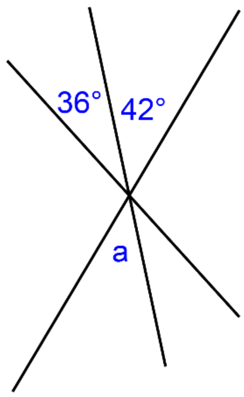
In the diagram below find the value of a.



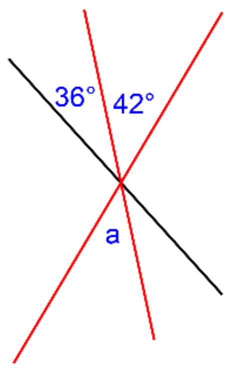
The sum of the angles on a straight line = 180° a + 2a + 3a = 180° 6a = 180° a = 30°

Finding vertically opposite angles

The diagram shows angles made between three straight lines. Find the value of a.

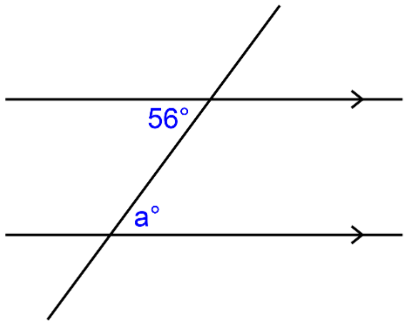


a and 42° are vertically opposite angles, as they are generated by the same straight lines shown in red in the diagram below, so are equal.

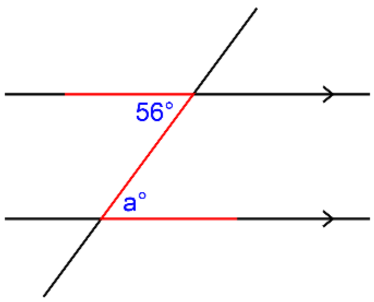


Alternate angles

Find the value of a in the diagram.

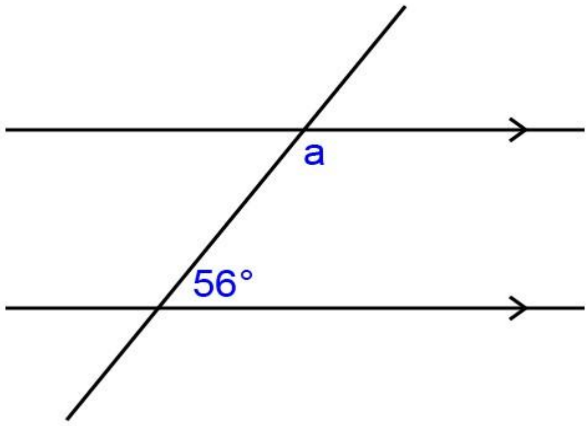


a is an alternate angle to the 56° shown. Alternate angles are the angles contained in the Z shapes formed by lines crossing parallel lines. The answer is 56°.

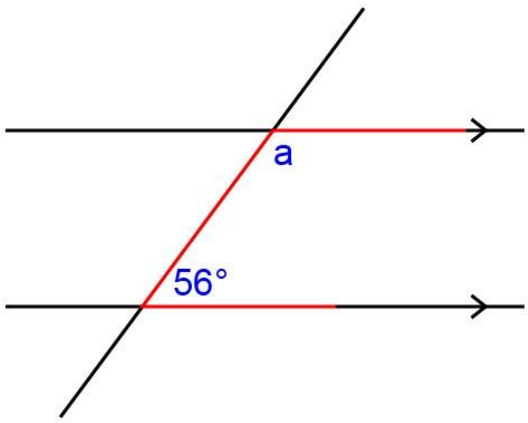


Co-interior or allied angles

Find the size of angle a in the diagram.



Angle a and 56°are co-interior or allied angles. They sit in the corners of the C shape shown in red.

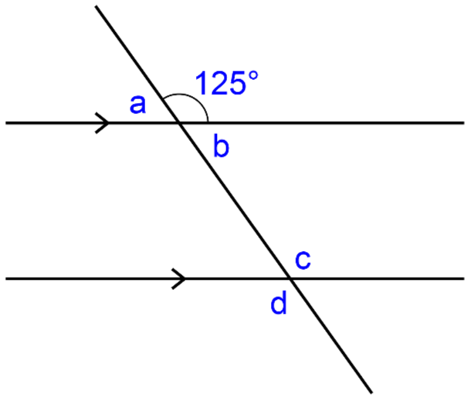


a + 56° = 180°

a = 180° − 56° = 124°

Finding multiple angles in parallel lines

Find the values of a, b, c and d in the diagram.



a + 125° = 180°   Angles on a straight line add to 180˚

So:

a = 55°

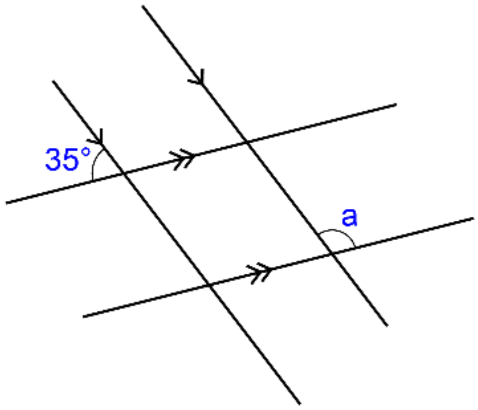
b and a are opposite angles so are equal c = 125°

c is a corresponding angle to the 125° given (or c and b are co-interior and c = 180° − b)

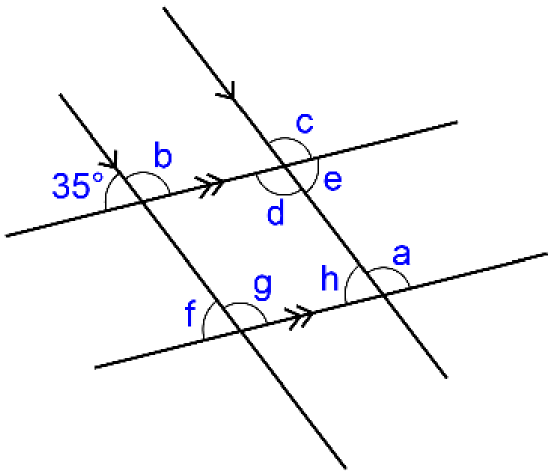
d = 125°   Vertically opposite to c

Combining angle facts

Find the size of the angle marked a in the diagram.



There are a large number of different ways of solving this; here are two possible methods but there are others.



Method 1

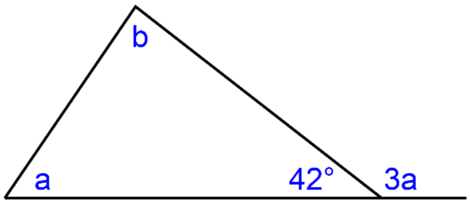
b = (180–35) = 145° Angles on a straight line add to 180˚
c = 145° Corresponding angle to b
a = 145° Corresponding angle to c

Method 2

f = 35° Corresponding angle to 35°
h = 35° Corresponding angle to f
a = (180–35) = 145° Angles on a straight line add to 180˚

Finding an unknown angle using exterior and interior angle properties of a triangle

Find the size of angles a and b in the diagram.



3a + 42° = 180°   Angles on a straight line add to 180˚

3a = 138°

a = 138 ÷ 3 = 46°

Either

Use the exterior angle property of a triangle:

a + b = 3a

b = 2a = 92°

or

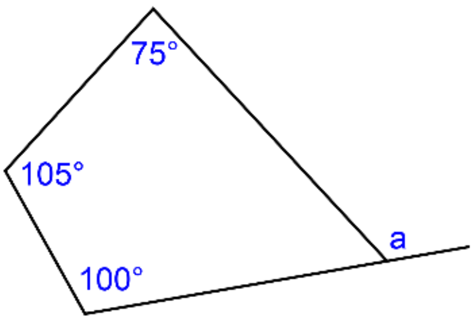
Use the angle sum of a triangle:

a + b + 42° = 180°

b = 180–42–46 = 92°

Finding an unknown angle using the angle sum of a quadrilateral

What is the size of angle a in the diagram?



The angle sum of a quadrilateral is 360°

The missing angle of the quadrilateral is 360–(75 + 105 + 100) = 80°

a + 80° = 180°   Angles on a straight line add to 180˚

a = 100°

Finding an unknown angle in a polygon

Three angles of a pentagon are 100°, 110° and 82°
The other two angles are equal.
What is the size of each of these two angles?

A pentagon is a 5-sided figure. If you join one of its vertices to all the other vertices, three triangles are formed (number of sides – 2).
The angle sum of a pentagon is $3 \times 180 = 540°(180(n–2))$
Subtract the three given angles $540–(100 + 110 + 82) = 248°$
Divide by 2
$248 \div 2 = 124°$

Finding interior angles of a regular polygon

What is the size of the interior angle of a 12-sided polygon (a dodecagon)?

Method 1

The polygon is regular, so all the angles are equal.
The sum of the exterior angles of any polygon is 360° so each exterior angle of a 12-sided figure is $360 \div 12 = 30°$
The interior angle is $180–30 = 150°$ Angles on a straight line add to 180˚

Method 2

The sum of the interior angles is $180(12–2) = 1800°$
Each interior angle is $1800 \div 12 = 150°$

Exterior angles of a polygon

The exterior angles of a pentagon are 45°, 88°, 70°, 30° and x°. Find the value of x.

The sum of the exterior angles of any polygon is 360°
$45 + 88 + 70 + 30 + x = 360$
$x = 360 - 233 = 127°$

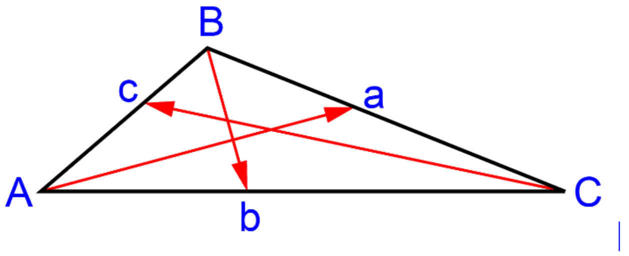
M5.3

Derive and apply the properties and definitions of special types of quadrilaterals, including square, rectangle, parallelogram, trapezium, kite and rhombus.

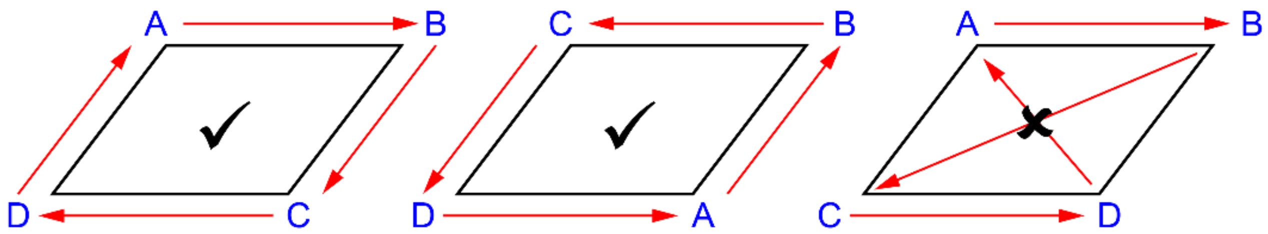
Derive and apply the properties and definitions of various types of triangle and other plane figures using appropriate language.

Labelling

In the triangle ABC, the capital letters A, B and C refer to the angles at the vertices, and the small letters a, b and c refer to the sides opposite those angles.

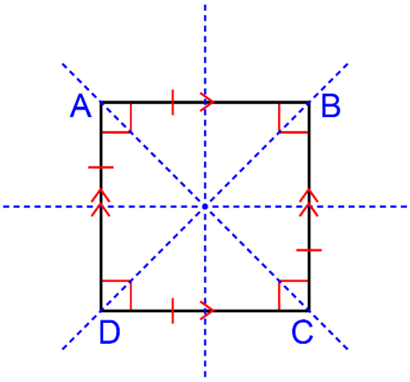


Quadrilaterals, and other plane figures, should be labelled consistently either clockwise or anti-clockwise. It does not matter which vertex you start labelling from.



Types of quadrilaterals

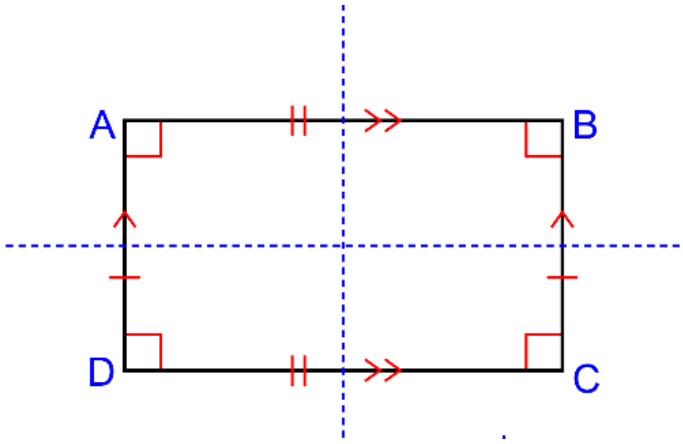
Square



A square is a regular quadrilateral.
It has two pairs of parallel sides and all its sides are equal and all interior angles are 90° It has 4 lines of symmetry and order of rotational symmetry 4.

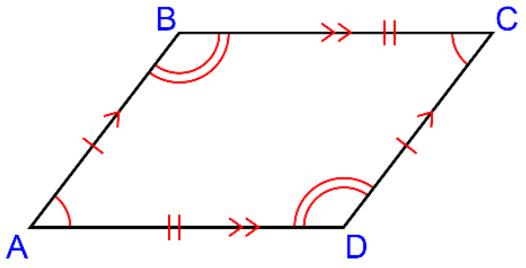
Notation:
The single arrows indicate that AB and DC are parallel. The double arrows indicate that DA is parallel to CB.
The single marks indicate that all 4 sides are equal in length.
The dotted lines are the 4 lines of symmetry.

Rectangle



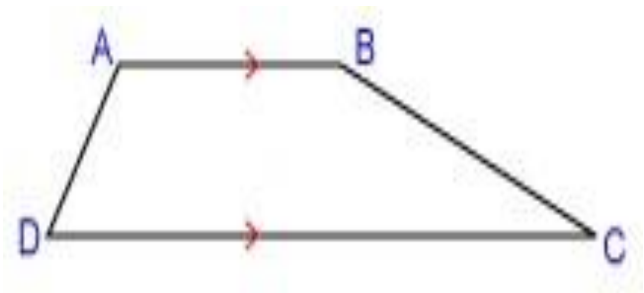
A rectangle has 2 pairs of parallel sides and all its interior angles are 90°
It usually has 2 lines of symmetry and order of rotational symmetry 2 unless it is a special type of rectangle i.e. a square.

Parallelogram



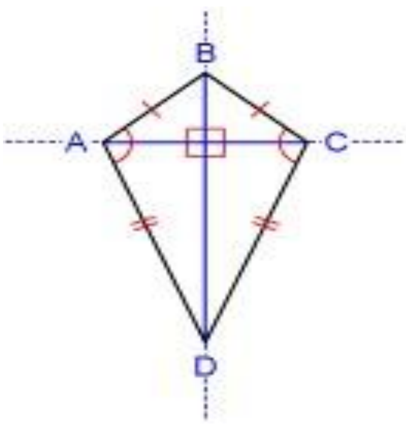
A parallelogram has 2 pairs of parallel sides.
Opposite sides are equal in length.
Opposite angles are equal and adjacent angles are supplementary.
It usually has no lines of symmetry and order of rotational symmetry 2, unless it is a special type of parallelogram i.e. a square, rhombus or rectangle.

Trapezium



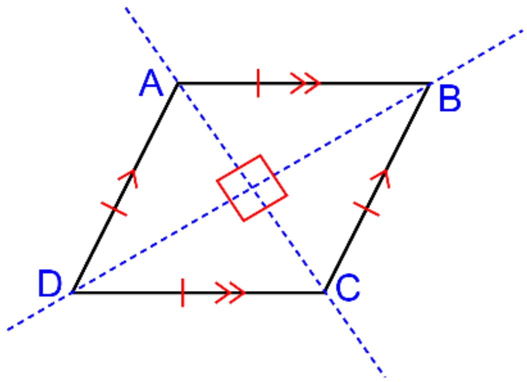
A trapezium has exactly one pair of parallel sides.
In most cases, it has no lines of symmetry.

Kite



A kite has 2 pairs of equal sides and only 1 line of symmetry.
It has one pair of opposite angles, which are equal.
The diagonals of a kite intersect at right angles.

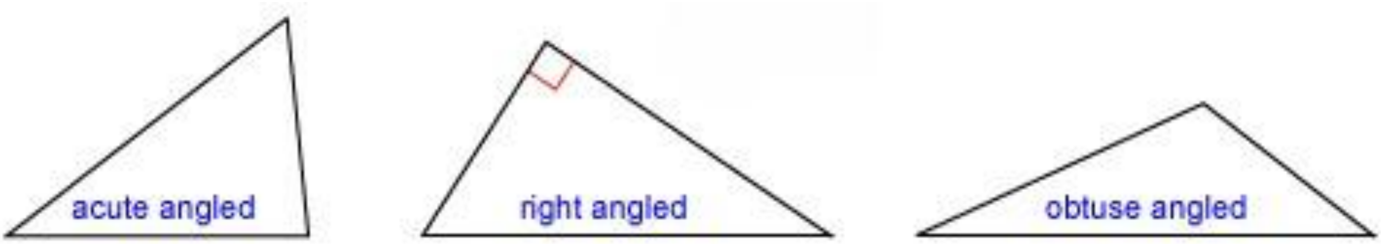
Rhombus



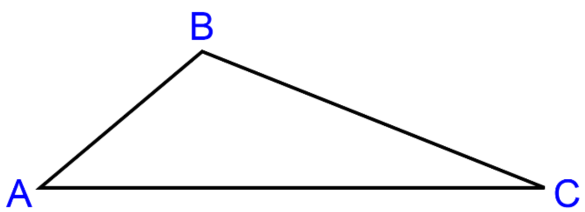
A rhombus has 2 pairs of parallel sides and all four sides are equal in length.
The diagonals of a rhombus bisect each other at right angles. It has 2 lines of symmetry and rotational symmetry order 2 unless it is a special type of rhombus (a square).

Types of triangle

An acute angled triangle has all of its angles less than 90°. A right-angled triangle has one right angle (90°). An obtuse angled triangle has an angle greater than 90°.

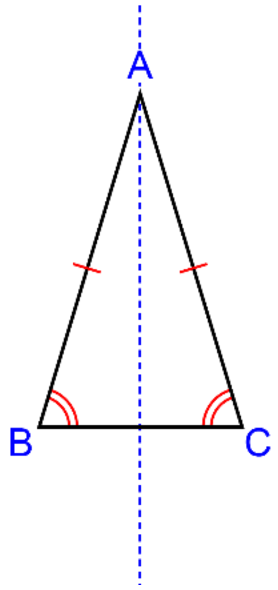


Scalene triangles



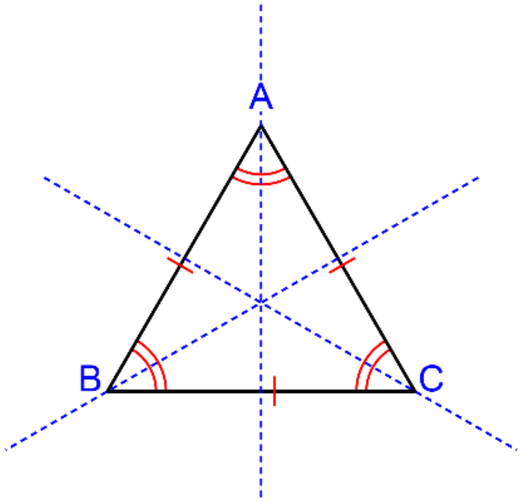
A scalene triangle has no equal sides and no right angles. It has no lines of symmetry.

Isosceles triangles



An isosceles triangle has 2 equal sides. It can be acute angled, right angled or obtuse angled.
It has one line of symmetry and no rotational symmetry.

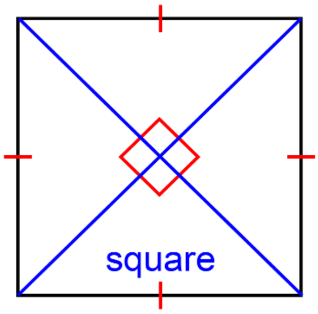
Equilateral triangles

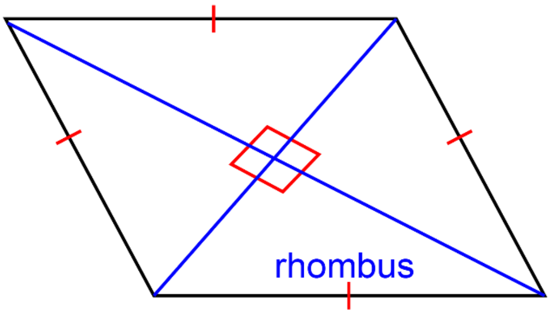


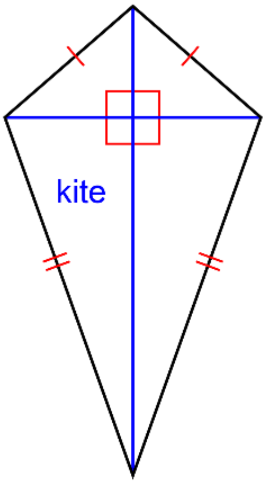
An equilateral triangle has 3 equal angles and three equal sides. It has 3 lines of symmetry and order of rotational symmetry 3.

Properties and definitions of special types of quadrilaterals

Name three types of quadrilaterals whose diagonals intersect at right angles.





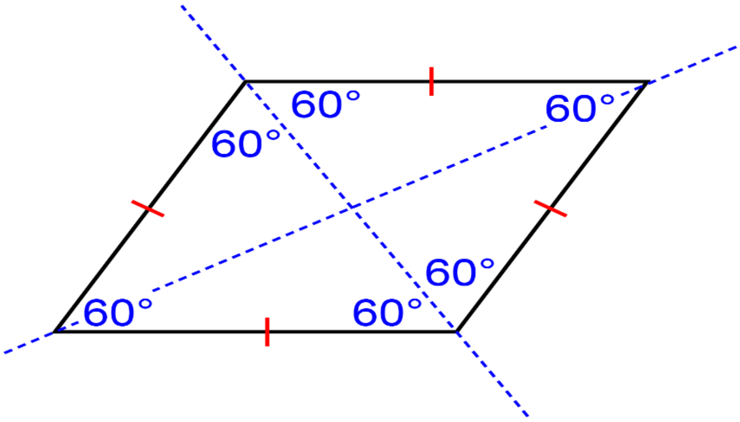


A delta or dart is a possible answer, but it is a special case of a kite.

Properties and definitions of special types of triangle and quadrilaterals

Two identical equilateral triangles are joined along one edge to form a quadrilateral. Give the name of the quadrilateral and describe its symmetries.

If two identical equilateral triangles are joined then the resulting quadrilateral has all 4 sides equal, so it is either a square or a rhombus. It has two opposite angles of 60° and two opposite angles of 120°, so it is a rhombus and not a square.
It has 2 lines of symmetry (shown), and order of rotational symmetry 2 as it would fit back onto itself twice in a 360° rotation about its centre.



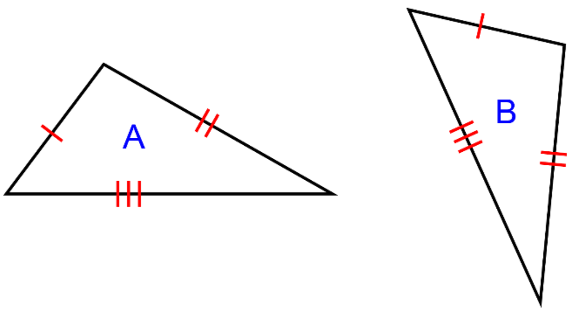
M5.4

Understand and use the basic congruence criteria for triangles (SSS, SAS, ASA, RHS).

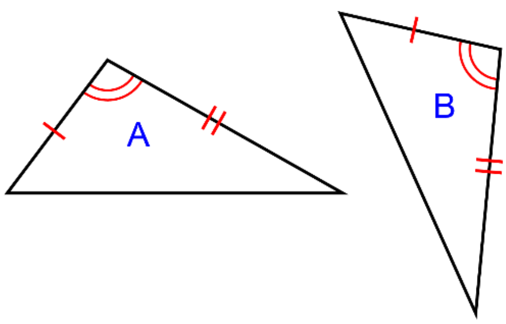
Definition

Two shapes are congruent if they are identical in shape and size.

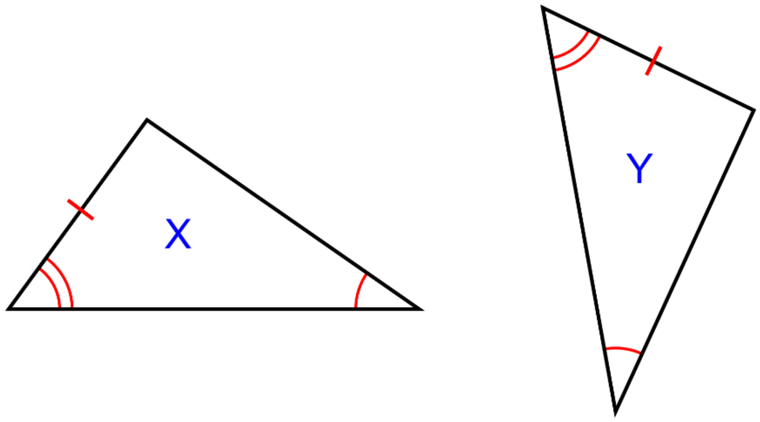
SSS (side, side, side): Two triangles A and B are congruent if the three sides of A are the same lengths as the three sides of B.



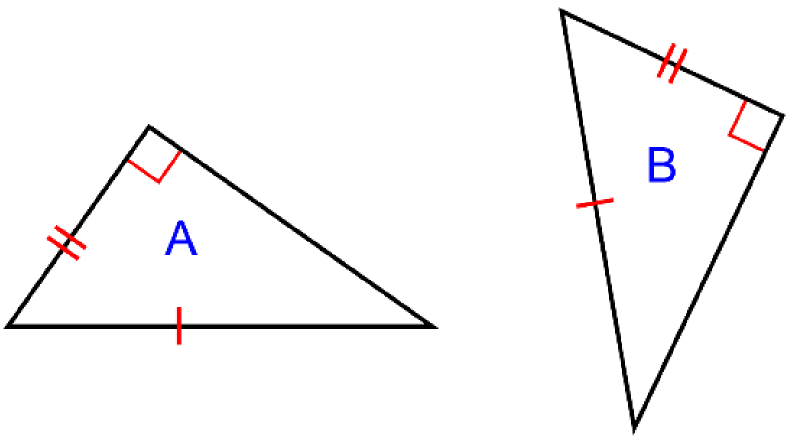
SAS (two sides and the angle included between them): Two triangles A and B are congruent if two sides and the angle between the two sides in shape A are the same as two sides and the angle between the two sides in shape B.



ASA (two angles and a corresponding side equal): Two triangles X and Y are congruent if two angles of X are the same size as two angles of Y, and a corresponding side of each triangle is equal.
Corresponding means that the sides are in the same place relative to the angles.
In triangles X and Y the equal sides correspond because they are both opposite the angle marked with a single arc.



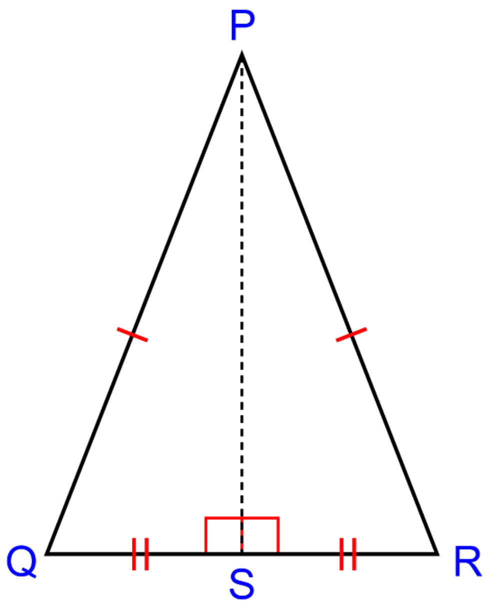
RHS (right angle, hypotenuse and side): Two triangles A and B are congruent if both triangles are right-angled, have the same length hypotenuse (the longest side of a right-angled triangle – the side opposite the right angle), and have one other side the same length.



SSS (side, side, side)

PQR is an isosceles triangle with PQ = PR
S is the midpoint of QR
Show that triangle PQS is congruent to triangle PRS

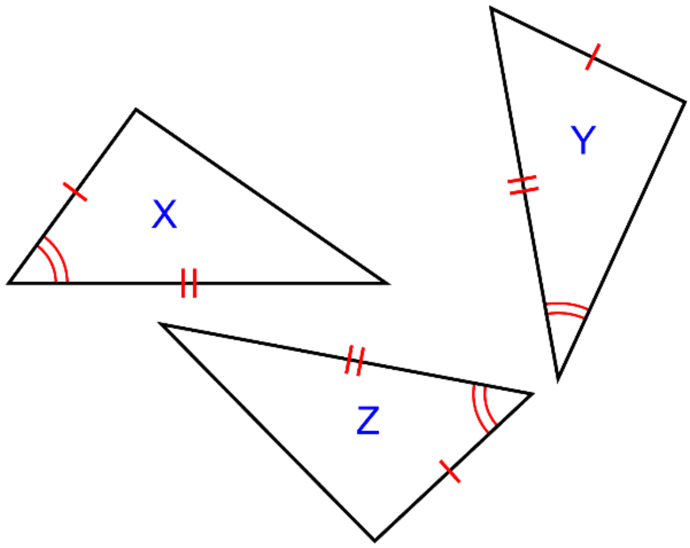
Always draw a diagram.



PQ = PR
PS is common.
QS = SR (given)
Therefore, triangle PQS is congruent to triangle PRS (SSS)

SAS (two sides and the angle included between them)

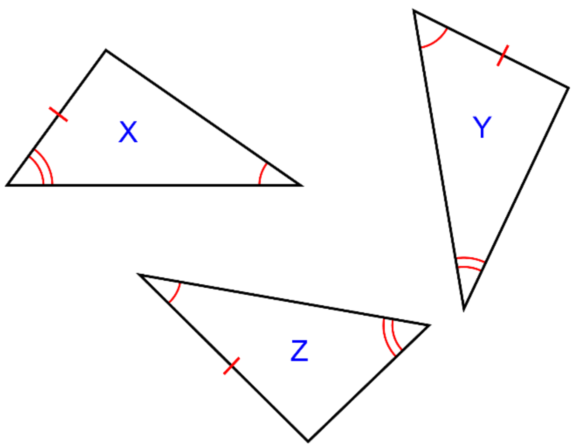
Which two of these three triangles (not drawn to scale) must be congruent?



Do not guess by looking at the shape of the triangles as they are not drawn to scale.
All three triangles have corresponding pairs of sides which are equal.
In triangles X and Z, the equal angles are included between the sides so they must be congruent but in triangle Y, the equal angle is not included between the corresponding equal sides.

ASA (two angles and a corresponding side equal)

Which two of these three triangles must be congruent?



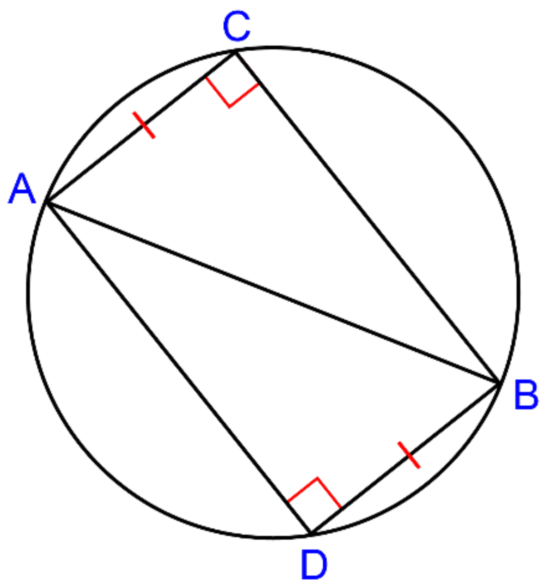
Do not guess by looking at the shape of the triangles as they are not drawn to scale.
All three triangles have two equal angles, so you need to check that the equal sides are corresponding.
The side with the single bar is opposite the angle with one arc in X, and opposite the side with 2 arcs in Y and Z, so Y and Z are congruent.

RHS (right angle, hypotenuse and side)

AB is the diameter of a circle S. C and D are points on the circumference of S and are on opposite sides of the diameter AB.

If AC = BD show that triangle ACB is congruent to triangle ADB

Always draw a diagram and label it clearly.



Angle ACB = angle ADB = 90°   (Angle in a semicircle)
The hypotenuse of triangle ACB = The hypotenuse of triangle ACB = AB
AC = BD   (given)
So, triangle ACB is congruent to triangle ADB (RHS)

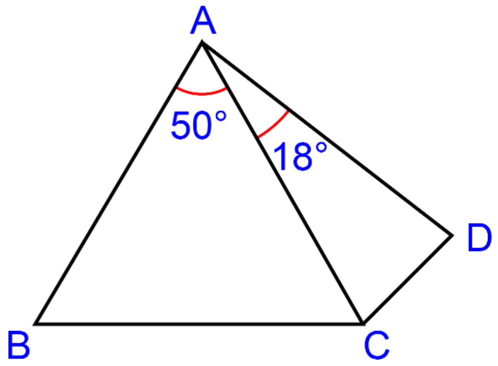
M5.5

Apply angle facts, triangle congruence, similarity, and properties of quadrilaterals to results about angles and sides.

You will need to recall the information from sections M5.1 to M5.4.

Properties of triangles

In the isosceles triangle ABC, AB = AC and angle BAC = 50°. Another isosceles triangle CAD is drawn in the same plane as triangle ABC, as in the diagram, with angle CAD = 18°

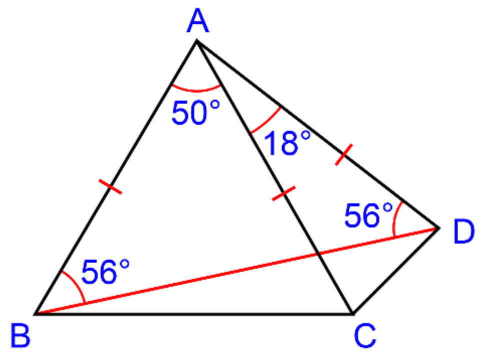


What is the size of angle DBC?

Mark all equal lengths onto the diagram: AB = AC = AD

This shows that triangle ABD is isosceles.

Join B and D to form the triangle DBC, which contains the angle DBC.



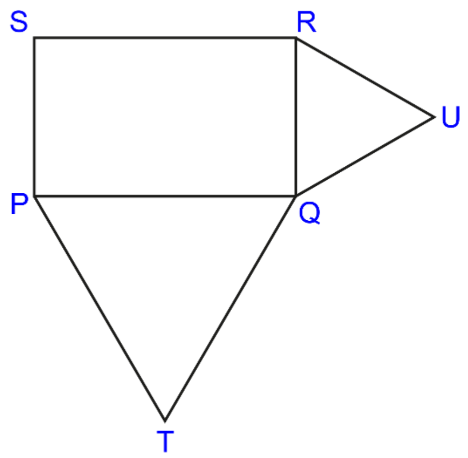
Angle ABC = angle ACB = $\frac{180-50}{2}$= 65° (base angles of isosceles triangle)

Angle BAD = 50° + 18° = 68°

Angle ABD = angle ADB = $\frac{180-68}{2}$ = 56°

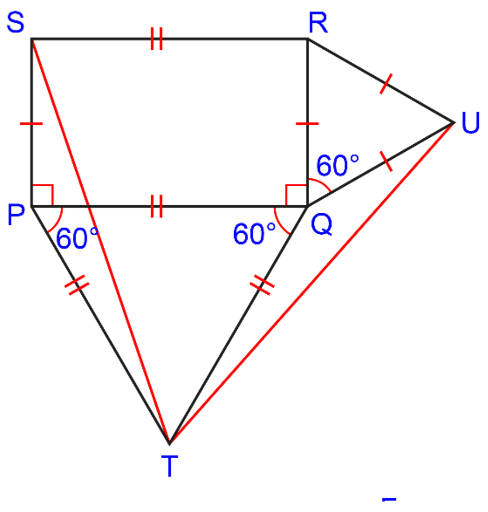
Angle DBC = 65° − 56° = 9°

Properties of triangles and special quadrilaterals



In the diagram, PQRS is a rectangle and PQT and QRU are equilateral triangles.
If the length of TS is 10 cm, what is the length of TU?

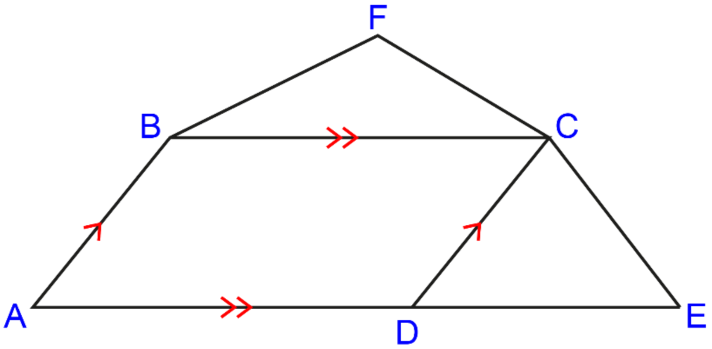
First, draw in the lines TS and TU on the diagram and mark equal sides.
SP = QR (opposite sides of a rectangle)
SP = QR = RU = QU (sides of an equilateral triangle)
Similarly, SR = PQ = PT = QT
In the triangles SPT and UQT:
SP = QU
PT = QT



Now check the angle included by these two sides.
Angle SPT = 90° + 60° = 150°
Angle TQU = 360° − (60° + 90° + 60°) = 360° − 210° = 150°
Triangle SPT is congruent to triangle UQT (SAS).
This means that the triangles are identical so TU = TS = 10 cm

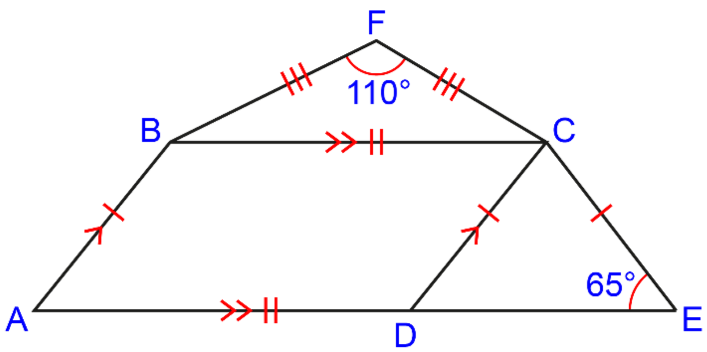
Properties of triangles and special quadrilaterals

ABCD is a parallelogram.
CDE is an isosceles triangle with CD = CE and ADE is a straight line.
Angle CED = 65°
Triangle BFC is isosceles with BF = FC
Angle BFC = 110°



What is the size of angle ABF?

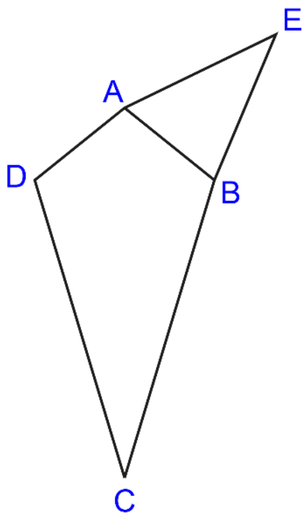
Mark the equal sides onto the diagram.
Put in the given angles.
Angle CDE = 65° (Base angles of an isosceles triangle are equal)
Angle ADC = 180° – 65° = 115° (Angles on a straight line add up to 180°)
Angle ABC = angle ADC = 115° (Opposite angles of a parallelogram are equal)
In triangle FBC, angle FBC = angle FCB (Base angles of an isosceles triangle)
Angle FBC = $\frac{180-110}{2}$ = 35°



Angle ABF = angle FBC + angle ABC = 35° + 115° = 150°

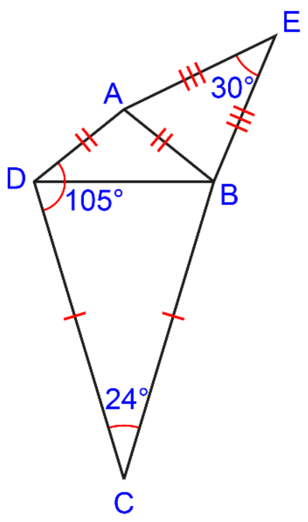
Properties of triangles and special quadrilaterals

ABCD is a kite.
ABE is an isosceles triangle with EA = EB, as shown in the diagram.



Angle ADB = 105°. Angle DCB = 24° and angle AEB = 30°
What is the size of angle DBE?

Mark all equal lengths and given angle sizes onto the diagram.
Angle ADC = angle ABC = 105° (symmetry of a kite)
Angle DAB = 360° − (105° + 105° + 24°) (angles of a quadrilateral add up to 360°)
Angle DAB = 126°
Angle DBA = $\frac{180-126}{2}$ = 27° (base angle of an isosceles triangle)
Angle EBA = $\frac{180-30}{2}$ = 75° (base angle of an isosceles triangle)



Angle DBE = angle DBA + angle ABE = 27° + 75° = 102°

M5.6

Identify, describe and construct congruent and similar shapes, including on coordinate axes, by considering rotation, reflection, translation and enlargement (including fractional and negative scale factors).

Describe the changes and invariance achieved by combinations of rotations, reflections and translations.

Describe translations as 2-dimensional vectors.

Note

When we write about transformations, the original shape is referred to as the object and the transformed shape as the image.

Points which stay in the same place under the transformation are called invariant points.

Rotation

In a 2-dimensional rotation, an object is rotated about a point in the plane. To describe a rotation fully, we need:

- the centre of rotation (the point about which the shape is being rotated)
- the angle of rotation (the angle turned through from object to image)
- the direction of the rotation (anticlockwise is the positive direction for a rotation).

The object and the image are the same size and shape and so are congruent to each other.

The invariant point is the centre of rotation, unless the angle of rotation is 0° or a multiple of 360° in which cases all points are invariant.

Reflection

In a 2-dimensional reflection, an object is reflected in a line.

The reflection is defined by the reflection line, sometimes referred to as the mirror line.

The object and image are the same shape and size, so they are congruent, but the object would have to be turned over to fit onto the image. This is sometimes referred to as opposite congruence.

The invariant points are the mirror line.

Translation and translation vectors

In a 2-dimensional translation, an object is moved without being reflected or rotated.

The translation is defined by the number of squares parallel to each of the axes the object is moved through to become the image.

The movement is written in the vector form $\binom{a}{b}$ where a is the number of squares moved in the x-direction and b is the number of squares moved in the y-direction. If the shape moves in the direction of increasing x values (to the right) then a is positive, and a is negative if moved in the opposite direction (to the left). If the shape moves in the direction of increasing y values (upwards), then b is positive, and b is negative if moved in the opposite direction (downwards).

The object and the image are the same size and shape and so are congruent to each other.

There are no invariant points unless the translation is $\binom{0}{0}$ in which case all points are invariant.

Enlargement

In a 2-dimensional enlargement of scale factor n about a given centre the image is n times the size of the object and is n times as far from the centre of the enlargement.

An enlargement is defined by the centre of enlargement and the scale factor.

A negative scale factor indicates that the object and image are on different sides of the centre of enlargement.

A scale factor between −1 and 1 indicates that the image is smaller than the object and nearer to the centre of enlargement.

Enlargement preserves shape but not size, so object and image are similar but not congruent (except if the scale factor is 1 or −1 in which case the shapes are both similar and congruent).

The centre of the enlargement is an invariant point unless the scale factor of the enlargement is 1 in which case all points are invariant.

Combinations of rotations, reflections and translations

Transformations can be combined by performing the transformations in the given order.

Any combination of rotation, reflection and translation will result in an image which is congruent to the object, as each transformation produces a congruent image.

Rotation

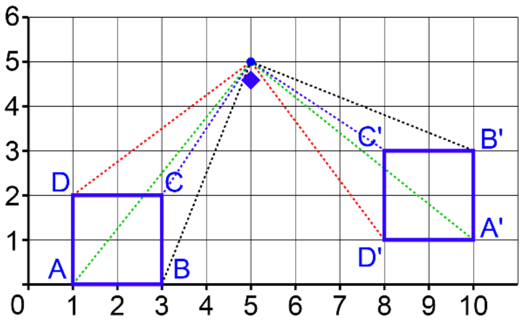
The square with vertices A (1, 0), B (3, 0), C (3, 2) and D (1, 2) is rotated through 90° anticlockwise about the point (5, 5) to form the square A′, B′, C′ and D′, where A′ is the image of A, B′ the image of B, C′ the image of C and D′ the image of D.

What are the coordinates of A′, B′, C′ and D′?

Always draw a diagram for transformation questions.

Method 1

Rotate each corner in turn through 90° about the point (5, 5) This can either be done with tracing paper: Trace the square, then pin the tracing paper to the point (5, 5) and rotate the paper through 90°. Label the corresponding points and join them to form a square.



or

With ruler and protractor:

join point A to the centre of rotation
measure the distance from A to the centre of rotation
use a protractor to measure the 90° rotation
draw the point A′ at the same distance from the centre of rotation as A.

Repeat for the other 3 vertices of the square.

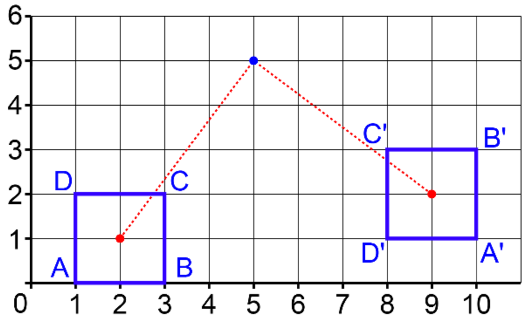
The image will be in a different place from the object and will have turned through 90° in the transformation.

or

Angles of 90° can be found by counting squares. As A is 5 squares down and 4 squares left from the centre of rotation, A′ is 4 squares down and 5 squares right.
A′ is at (10, 1), B′ is at (10, 3), C′ is at (8, 3) and D′ is at (8, 1).

Method 2 – useful for squares and circles

Rotate only the centre of the square through 90° about the point (5, 5).
Rotate the square ABCD through 90° and place it around the image of the centre.



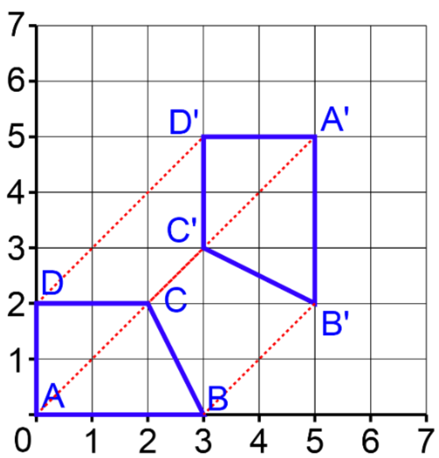
Reflection

The trapezium with vertices A (0, 0), B (3, 0), C (2, 2) and D (0, 2) is reflected in the line x + y = 5 to form the trapezium A′, B′, C′ and D′ where A′ is the image of A, B′ the image of B, C′ the image of C and D′ the image of D.

What are the coordinates of A′, B′, C′ and D′?

Always draw a diagram for transformation questions.

Draw the trapezium given and label the vertices.
Draw the mirror line given.
Reflect each vertex of the object in the mirror line by drawing a line from the point at right angles to the mirror line and extending it. The image and the object are the same distance from the mirror line.
Join the points to check that the image and object are congruent.
A′ is (5, 5), B′ is (5, 2), C′ is (3, 3) and D′ is (3, 5).



Translation

The trapezium with vertices A (1, 6), B (4, 6), C (3, 8) and D (1, 8) is translated 6 squares to the right and 4 squares down to form the trapezium A′, B′, C′ and D′ where A′ is the image of A, B′ the image of B, C′ the image of C and D′ the image of D.

What are the coordinates of A′, B′, C′ and D′?

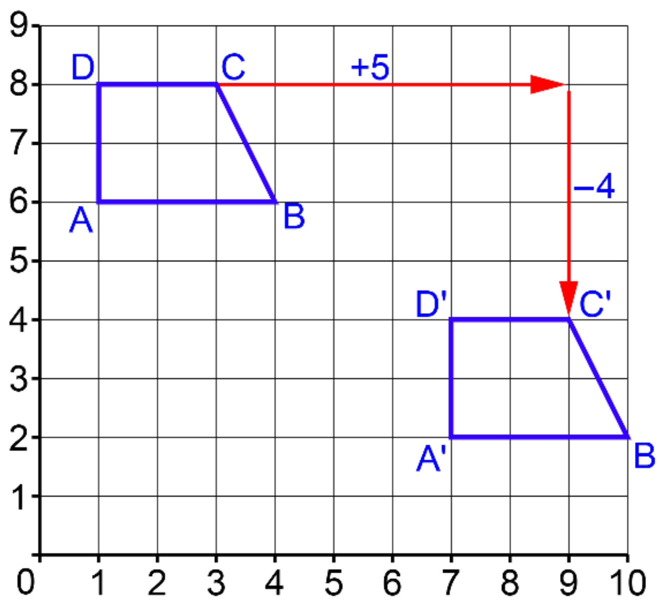
Write this transformation as a translation vector.

Always draw a diagram for transformation questions.

Draw the trapezium given and label the vertices.

Move each individual vertex 6 squares to the right and 4 squares down. The movement of vertex C is shown on the diagram.

Label the image vertices and join them to check that the object and image are congruent.
A′ is (7, 2), B′ is (10, 2), C′ is (9, 4) and D′ is (7, 4)



A 2-dimensional translation vector is written in a single vertical column with 2 rows.

The top number gives the number of squares moved in the x-direction with translation to the right as positive and left as negative.

The bottom number gives the number of squares moved in the y-direction with translation up as positive and down as negative.

This translation is 6 squares right and 4 squares down so is written as $\binom{6}{-4}$

Enlargement

Draw the image of the triangle with vertices A (2, 2), B (4, 2) and C (2, 6) after enlargement about the point P (0, 4) with:

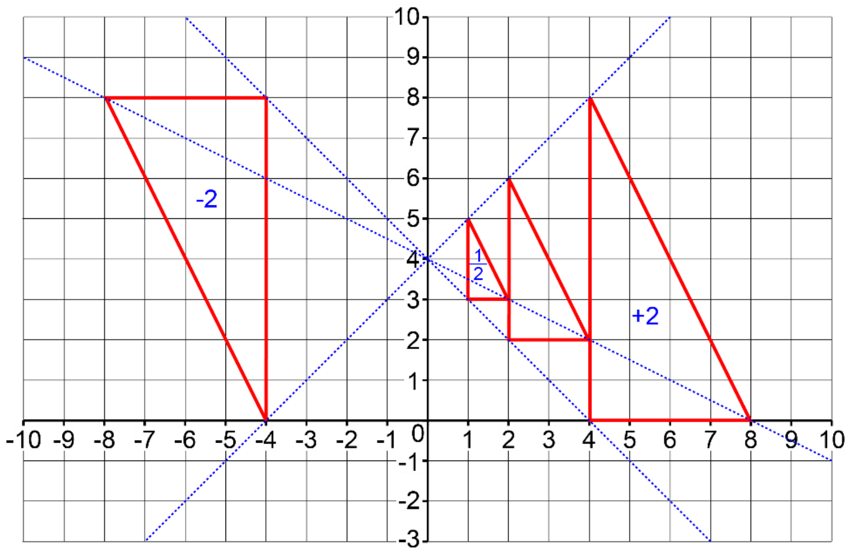
scale factor 2
scale factor $\frac{1}{2}$
scale factor -2

Always draw a diagram for transformation questions.

Draw the object triangle and the centre of the enlargement.

Join P to the vertices of the object and extend the lines – forward for positive scale factor enlargements, and backwards through the P for negative enlargements.

Measure the distance from P to the point A and multiply it by 2, the scale factor. Measure this distance from P along the line PA and mark the image of A. Repeat for B and C. Join the 3 points and the result is a triangle twice as far from P as ABC and with sides twice as long.



Or: Use translation vectors. PA = $\binom{2}{2}$ so OA′ = OP + 2PA = $\binom{0}{4}+\binom{4}{4}=\binom{4}{8}$ giving the coordinates of A′ as (4, 8). Repeat for B and C. This method can also be used for b) and c).
Measure the distance from P to the point A and multiply it by $\frac{1}{2}$, the scale factor. Measure this distance from P along the line PA and mark the image of A. Repeat for B and C. Join the 3 points and the result is a triangle half as far from P as ABC and with sides half as long.
Measure the distance from P to the point A and multiply it by 2, the scale factor. As the scale factor is negative, measure this distance from P in the opposite direction to PA, and mark the image of A. Repeat for B and C. Join the 3 points, and the result is a triangle twice as far from P as ABC, and with sides twice as long but on the opposite side of P from the object, and upside down.
Or: Find the scale factor +2 enlargement and rotate it through 180° about the centre of enlargement to get the scale factor –2 enlargement.

Combined transformations and invariance

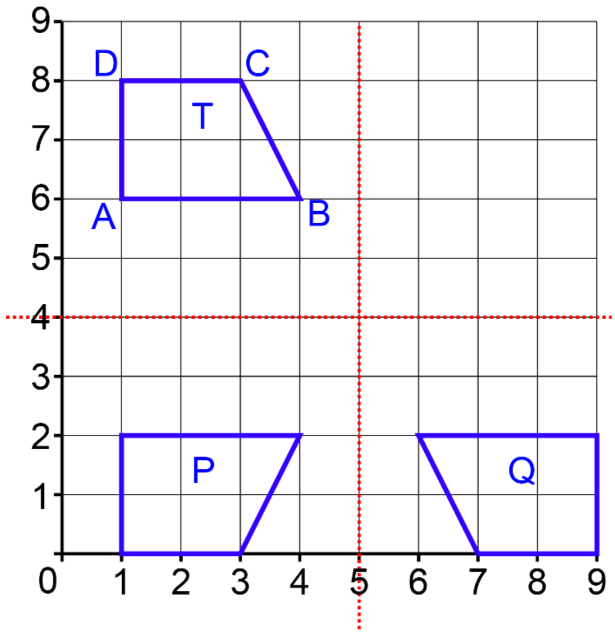
The trapezium, T, with vertices A (1, 6), B (4, 6), C (3, 8) and D (1, 8) is reflected in the line y = 4 to form the trapezium P, and then reflected in the line x = 5 to form the trapezium Q.

Which single transformation maps T to Q?

Which point(s) is (are) invariant in the combined transformation?

Always draw a diagram for transformation questions.

Do the reflections in the order given – it is not vital when dealing with 2 reflections, but generally changing the order of the transformations will give a different, and incorrect, result.
If the single transformation from T to Q was to be a reflection, then it would have to be in a diagonal mirror line. However, attempting to reflect T in this way would not give us a shape in the correct orientation (B′ would be 'pointing upwards').



It is a congruent shape, so it is either a translation or a rotation – in this case a rotation.

The diagram shows that T has been rotated through 180° to form Q, and that the centre of rotation is at (5, 4).

The invariant point is the centre of rotation, (5, 4), as it is the only point that does not change place under the transformation.

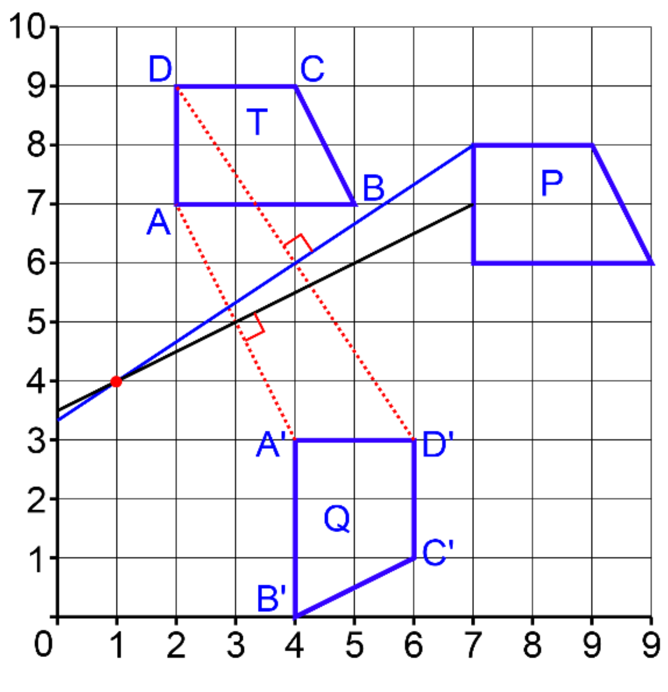
Combined transformations and invariance

The trapezium, T, with vertices A (2, 7), B (5, 7), C (4, 9) and D (2, 9) is translated with the translation vector $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ to form the trapezium P. P is then rotated through 90°clockwise about the point (4, 6) to form the trapezium Q.

Describe exactly the single transformation which maps T to Q.

Which point(s) is (are) invariant in the combined transformation?

Construct P and Q as in previous examples.
By inspection, Q is a rotation of T through 90° clockwise.
To find the centre of rotation:
Join a pair of corresponding vertices e.g. A and its image A'.
Construct the perpendicular bisector of AA'.
Join another pair of corresponding points e.g. D and D'.
Construct the perpendicular bisector of DD'.
Where the perpendicular bisectors meet is the centre of the rotation, in this case (1, 4).
The invariant point is the centre of this rotation: (1, 4).



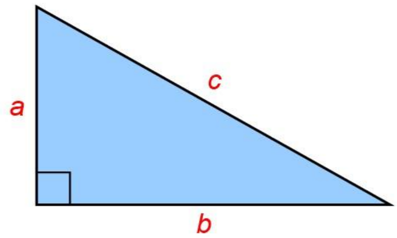
M5.7

Know and use the formula for Pythagoras' theorem: $a^2 + b^2 = c^2$

Use Pythagoras' theorem in both 2 and 3 dimensions.

Pythagoras' theorem in 2 dimensions

Pythagoras' theorem applies to right-angled triangles, and it says:



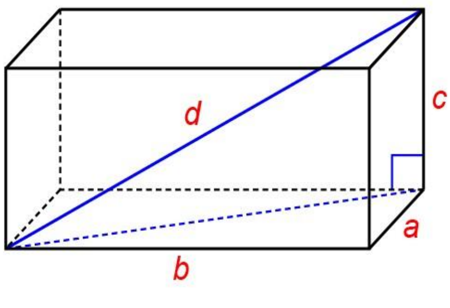
$$\mathbf{a^2 + b^2 = c^2}$$

You should be able to:

- find a missing side when we know the other two sides
- find a distance between two points given as coordinates
- identify whether or not a triangle is right angled
- find a missing length in other shapes containing right angles.

Pythagoras' theorem in 3 dimensions

Using Pythagoras' theorem in 3 dimensions we can find a missing length.
For example, we can find the length of the diagonal in a cuboid.



$$\boldsymbol{a^2 + b^2 + c^2 = d^2}$$

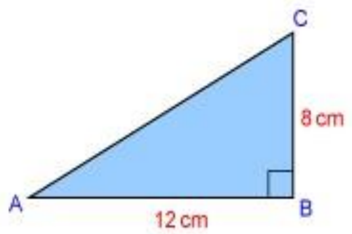
You should be able to:

- find missing lengths in 3-dimensional shapes
- solve problems involving Pythagoras' theorem in 2 or 3 dimensions (including using surds).

Find a missing side when we know the other two sides

Find the length of the hypotenuse (the longest side in a right-angled triangle; this is opposite the right angle).

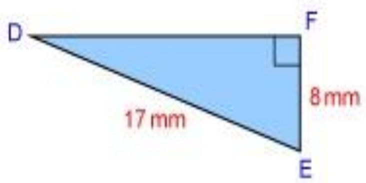
Find AC



$AC^2 = 8^2 + 12^2$
$AC^2 = 208$
AC = $\sqrt{208}$
AC = $4\sqrt{13}$ cm

Find a missing side when we know the other two sides

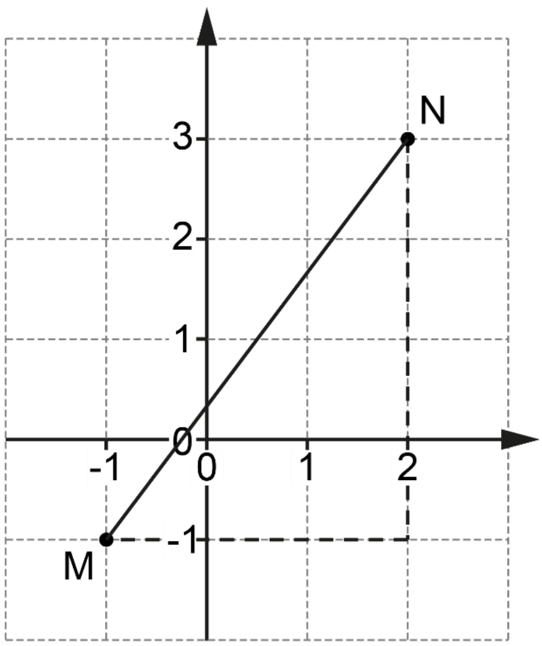
Find DF



$17^2 = 8^2 + DF^2$
$DF^2 = 17^2 – 8^2$
DF = $\sqrt{225}$
DF = 15

Find a distance between two points given as coordinates

M is the point (–1, –1),
N is the point (2, 3).
Find the length of the line MN.



Draw a sketch and form a right-angled triangle.

Find the horizontal and vertical distances which are 3 and 4.

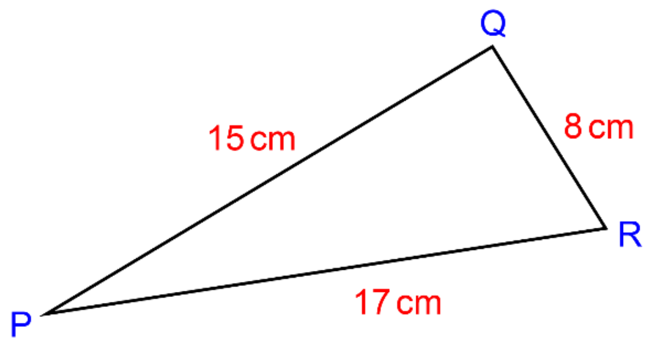
$MN^2 = 3^2 + 4^2$

$MN^2 = 25$

$MN = \sqrt{25}$

$MN = 5$

Identify whether or not a triangle is right-angled



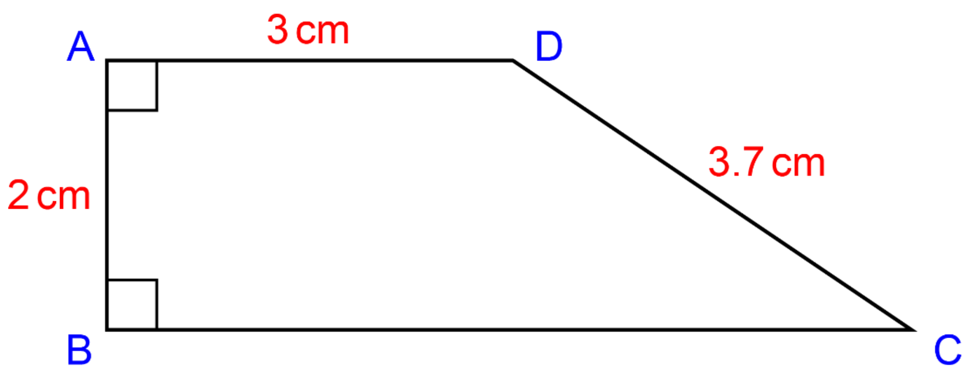
If Pythagoras' theorem applies, then triangle PQR is right-angled.
So, we need to check whether Pythagoras' theorem applies.
Assuming that the longest side (17) is the hypotenuse, we need to check if 17 squared is equal to 15 squared plus 8 squared:

$PQ^2 + QR^2 = 15^2 + 8^2 = 225 + 64 = 289$ and $PR^2 = 17^2 = 289$

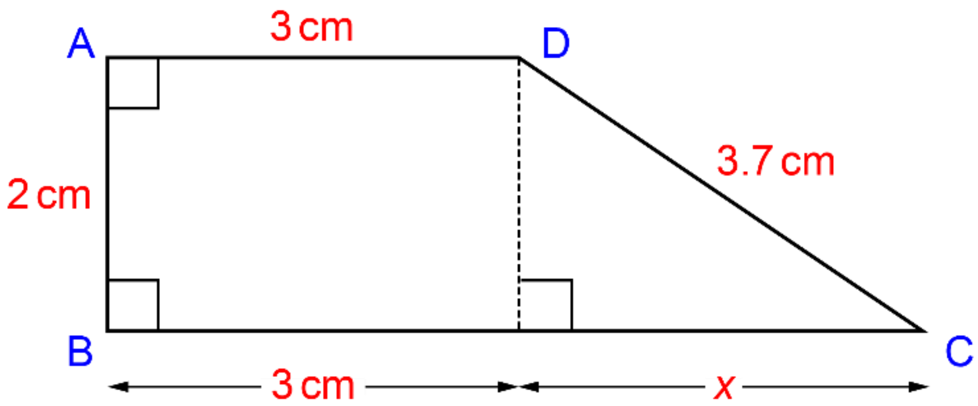
So $PQ^2 + QR^2 = PR^2$ and Pythagoras' theorem applies and the triangle is right-angled.

Find a missing length in other shapes containing right angles

Find the perimeter of the trapezium ABCD.



You will need to form the right-angled triangle.



$72 = 22 + x^2$

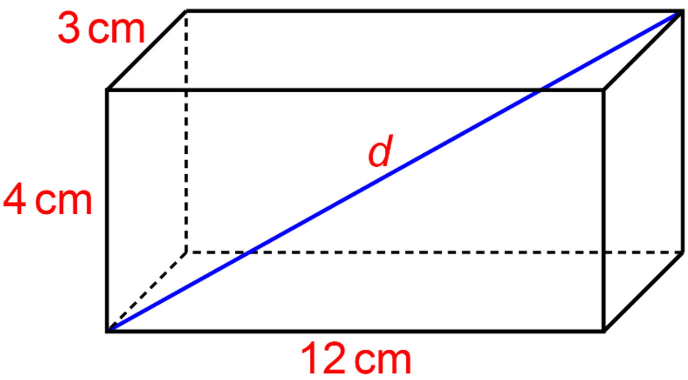
$x^2 = 7^2 – 2^2$

$x = \sqrt{45} = 3\sqrt{5}$

Perimeter = $3 + 2 + 3 + 3\sqrt{5} + 7 = 15 + 3\sqrt{5}$ cm

Find missing lengths in 3-dimensional shapes

Find d



Method 1

Using the formula $d^2 = a^2 + b^2 + c^2$ $d^2 = 3^2 + 4^2 + 12^2$

$d^2 = 169$
$d = \sqrt{169}$
d = 13 cm

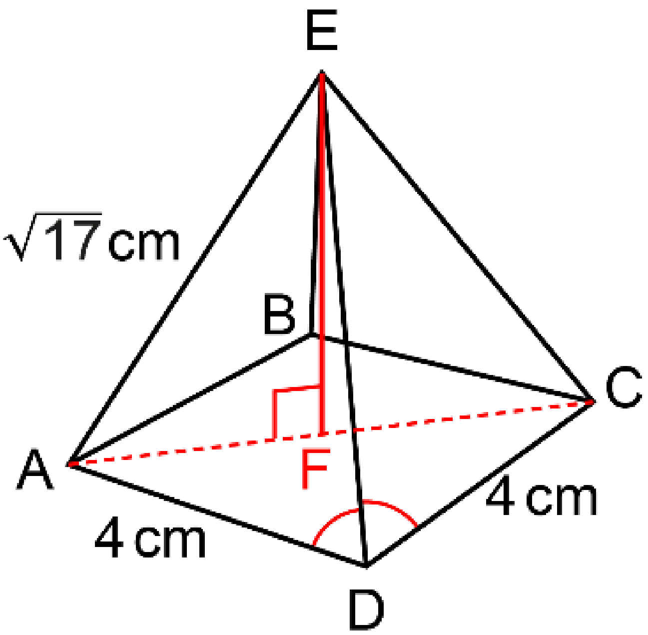
Method 2

Using 2D Pythagoras' theorem twice: $a^2 = 3^2 + 12^2$ where a is the diagonal of the base

$a^2 = 153$
$d^2 = a^2 + 4^2$
$d^2 = 153 + 16$
$d^2 = 169$
$d = \sqrt{169}$
d = 13 cm

Solve problems involving Pythagoras' theorem in 2 or 3 dimensions (including using surds)

The diagram shows a square based pyramid.



F is the centre of the square and E is vertically above F.
Find FE

Using Pythagoras' theorem in triangle ACD:

$AC^2 = 4^2 + 4^2 = 32$

$AC = \sqrt{32} = 4\sqrt{2}$

$AF = 2\sqrt{2}$

Using Pythagoras' theorem in triangle AFE:

$AE^2 = AF^2 + FE^2$

$(\sqrt{17})^2 = (2\sqrt{2})^2 + FE^2$

$17 = 8 + FE^2$

$FE = \sqrt{17 - 8} = 3$ cm

M5.8

Identify and use conventional circle terms: *centre*, *radius*, *chord*, *diameter*, *circumference*, *tangent*, *arc*, *sector* and *segment* (including the use of the terms *minor* and *major* for arcs, sectors and segments).

Diagram 1

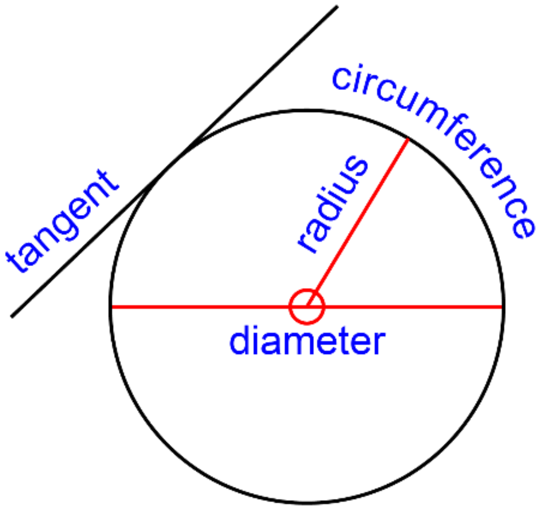


diagram 1

The circumference of a circle (the black outline on the diagram) is the distance around the outside of the circle.

The centre of a circle (O on the diagram) is the point which is the same distance from every point on the circumference. It is the point where the lines of symmetry meet.

A line segment from the centre to the circumference of the circle is the radius of the circle. The term radius is also used to refer to the length of this line.

The diameter of a circle is a line segment that passes through the centre of the circle and has endpoints at the circle.

The diameter is twice the radius.

A tangent is a line which touches the circle but does not cut it.

Diagram 2

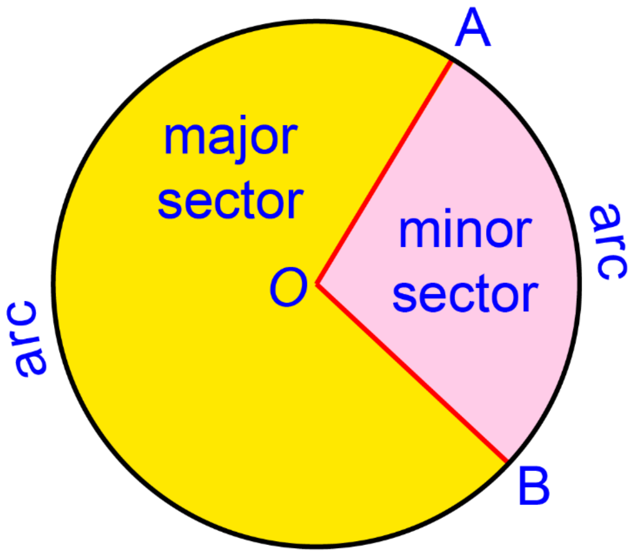


diagram 2

An arc is a part of the circumference of a circle. A minor arc measures less than a semicircle. A major arc measures more than a semicircle.

A sector is a part of a circle bounded by two radii and an arc.

Two radii will cut a circle into two sectors – if they are different sizes then the larger one is the major sector and the smaller is the minor sector.

Diagram 3

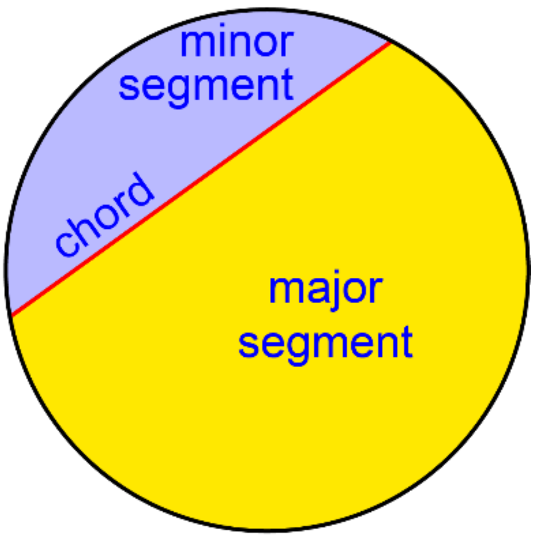
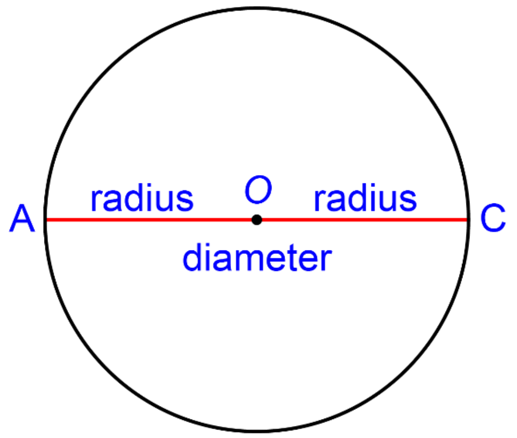


diagram 3

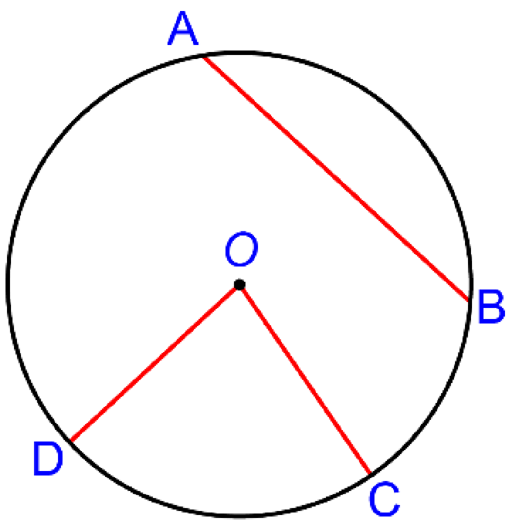
A chord is a straight line joining two points on the circumference of a circle. It cuts the circle into two segments. If the two segments are different in size then the larger one is the major segment and the smaller is the minor segment.

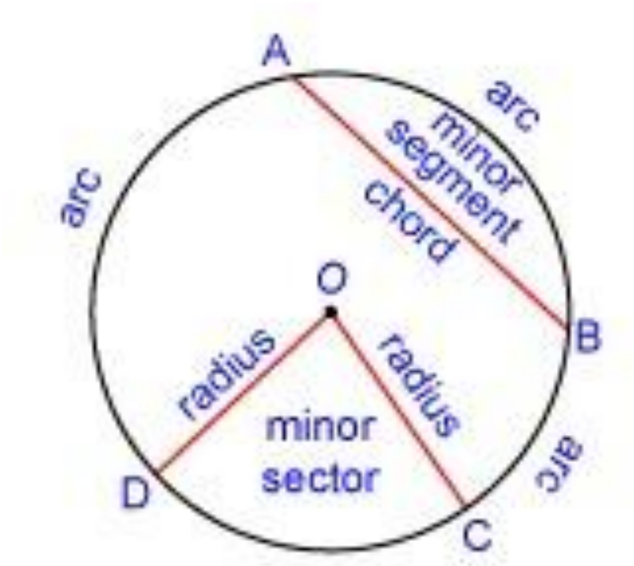
What is the relationship between the radius and the diameter of a circle?

The diameter AC is made up of the two radii OA and OB so the length of a diameter is twice the length of a radius.



Put the labels in the correct place on the diagram:





M5.9

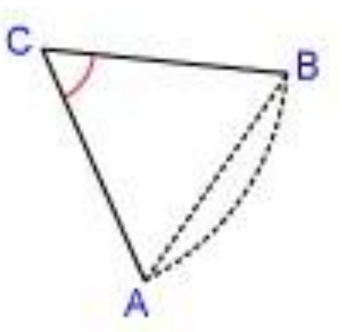
Apply the standard circle theorems concerning angles, radii, tangents and chords, and use them to prove related results:

a. angle subtended at the centre is twice the angle subtended at the circumference
b. angle in a semicircle is 90°
c. angles in the same segment are equal
d. angle between a tangent and a chord (alternate segment theorem)
e. angle between a radius and a tangent is 90°
f. properties of cyclic quadrilaterals

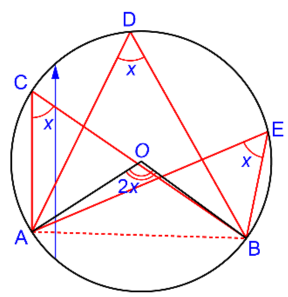
The angle subtended at the centre of a circle by a chord

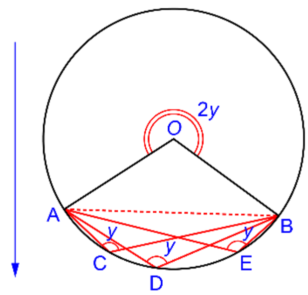
In the diagrams, angle ACB is the angle subtended at the point C by the arc AB or the points A and B or the line segment AB.

An angle can be subtended by two points or a line segment or a curve or an object.



The angle subtended at the centre of a circle by a chord is twice the angle subtended at the circumference by the same chord.



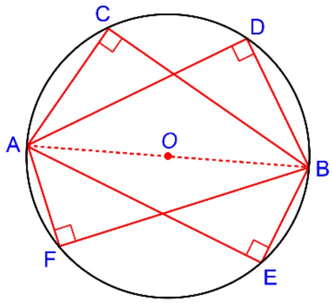


The angles in the diagrams are all based on the chord AB. The angle at the centre of the circle subtended by the chord AB – angle AOB – is twice the angle at the circumference of the circle subtended by the chord AB.

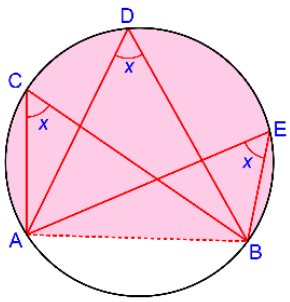
Note: Direction is important. A and B are the same points on the same circle in both the diagrams above, but one is looking at the obtuse angle AOB and the other is looking at the reflex angle.

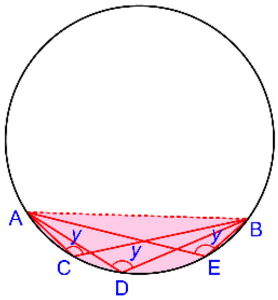
The angle in a semicircle is 90˚

AB is a diameter cutting the circle into two semicircles. The angle at the centre is 180°, so the angles at the circumference are all 90°.



Angles in the same segment are equal

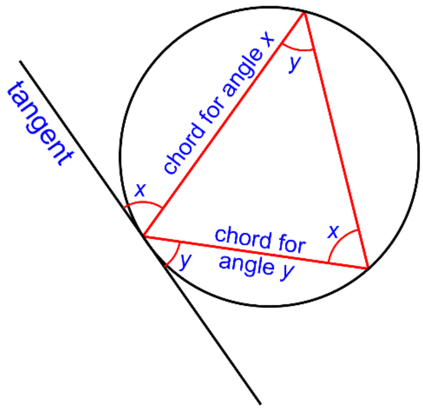




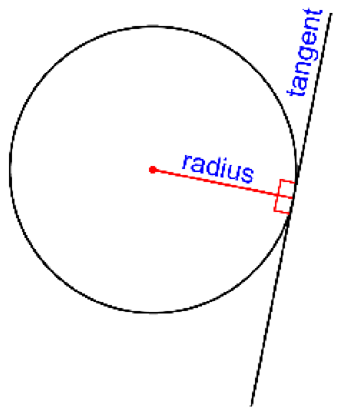
The segment is defined by the chord AB. The chord AB is the same in both diagrams, but the angles x and y are not equal as they are in different segments.

Alternate segment theorem

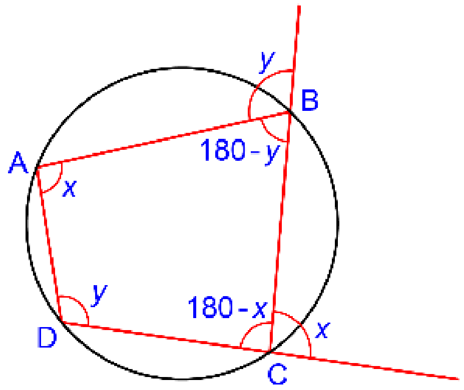
The angle between tangent and chord is equal to the angle subtended by that chord (alternate segment theorem).



Angle between radius and tangent is 90°



Properties of cyclic quadrilaterals

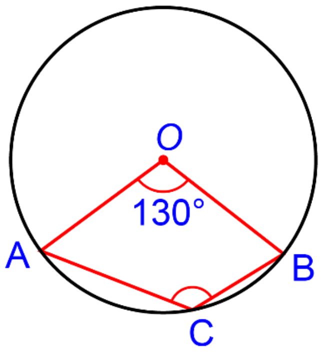


ABCD is a cyclic quadrilateral.
Both pairs of interior opposite angles add up to 180°.
The exterior angle is equal to the interior opposite angle.

Note: Most questions on this topic require the use of more than one circle theorem. The last example below is one such question.

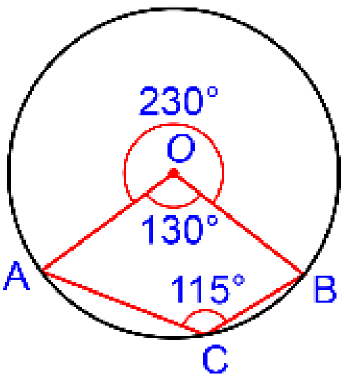
Example

The angle subtended at the centre of a circle by a chord
A, B and C are points on the circumference of a circle, centre O.
The obtuse angle AOB is 130°, as shown on the diagram.



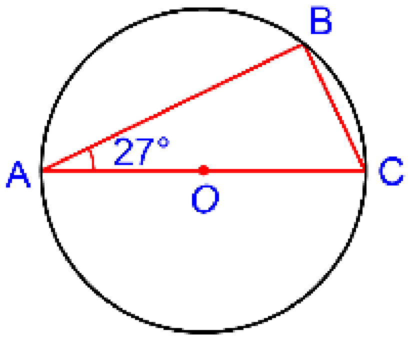
What is the size of the marked angle ACB?

The angle at the centre which is twice angle ACB is not the obtuse angle of 130° at O, but the reflex angle which needs to be calculated first.
The reflex angle is 360° − 130° = 230°
230° is twice the angle ACB
Angle ACB = 230° ÷ 2 = 115°



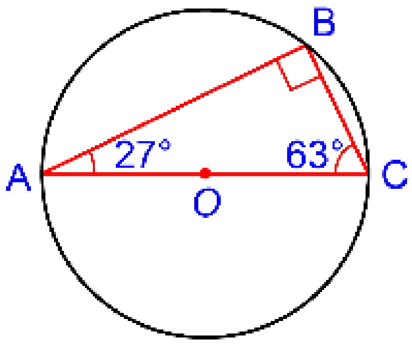
Angle in a semicircle

ABC is a triangle inscribed in a circle centre O.
AC is a diameter of the circle.
Angle BAC = 27°



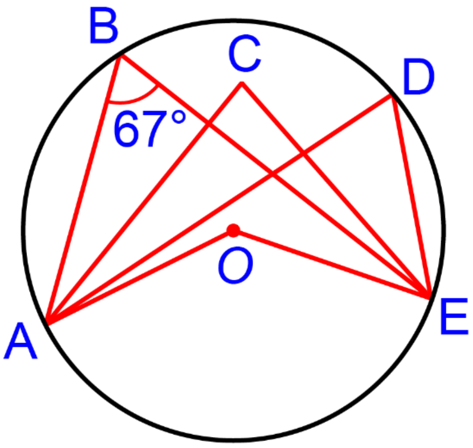
What is the size of angle BCA?

As AOC is a diameter it splits the circle into two semicircles.
The angle in a semicircle is a right angle so angle ABC = 90°
The angles of a triangle total 180° so angle BCA = 180° − 90° − 27° = 63°



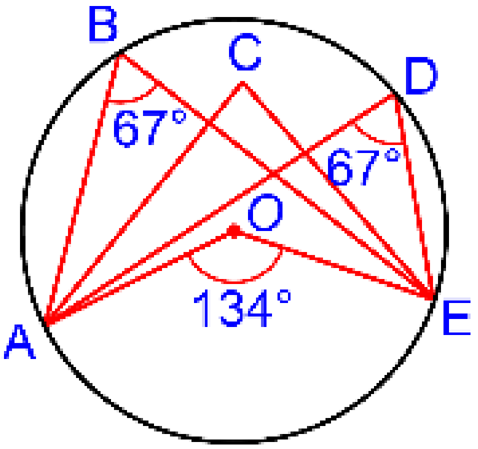
Using more than one circle theorem

A circle centre O is shown in the diagram.
Angle ABE = 67°



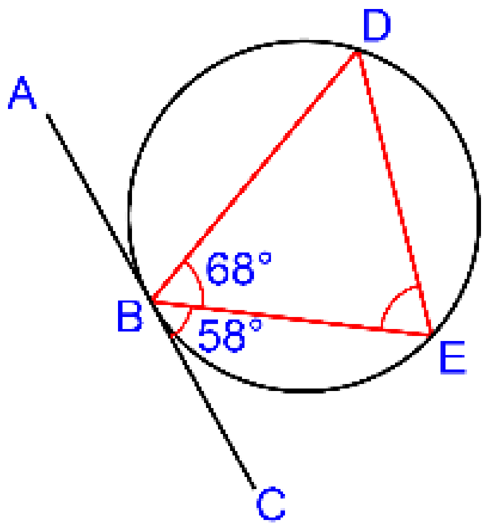
Which other angle in the diagram must be 67°?
What is the size of angle AOE?

Angle ABE is in the segment defined by chord AE. The other angle subtended at the circumference of the circle subtended by the chord AE is angle ADE.
Note: Angle ACE is also subtended by chord AE, but not at the circumference, so it is not equal to angle ABE.
Angle AOE is the angle at the centre subtended by AE so is twice angle ABE, so angle
AOE = 2 × 67° = 134°



Alternate segment theorem

BDE is a triangle inscribed in a circle. ABC is a tangent to the circle.



Angle BDE = 68° and angle EBC = 58°

What is the size of angle BED?

Method 1

ABC is a straight line so angle ABD = 180° − 68° − 58° = 54°

Angle ABD is between the tangent and the chord BD. The alternate, or other, segment defined by the chord BD is the segment on the opposite side of the chord from the angle ABD. The angle in this segment is the angle BED so:

Angle ABD = angle BED = 54°

Method 2

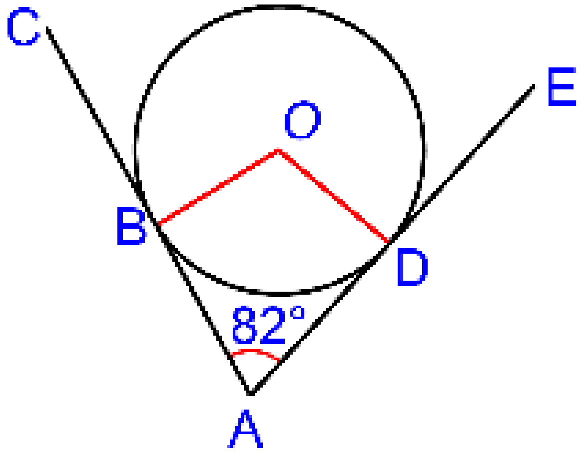
Angle CBE is between the tangent and the chord BE. The alternate segment is the one on the other side of the chord BE from the angle CBE.

The angle subtended by the chord BE on the alternate segment is angle BDE = 58°

In the triangle BDE, the angle BED = 180° − (68° + 58°) = 54°

Angle between a tangent and a radius

ABC and ADE are tangents at the points B and D to a circle centre O.
Angle BAD = 82°

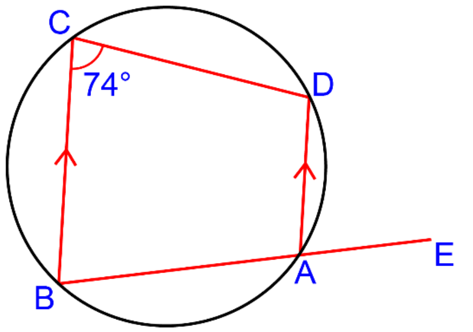


What is the size of angle BOD?

Angles OBA and ODA are both angles between a radius and a tangent so:
Angle OBA = angle ODA = 90°
OBAD is a quadrilateral, so the sum of its angles is 360°
Angle BOD = 360° − (90° + 90° + 82°) = 98°

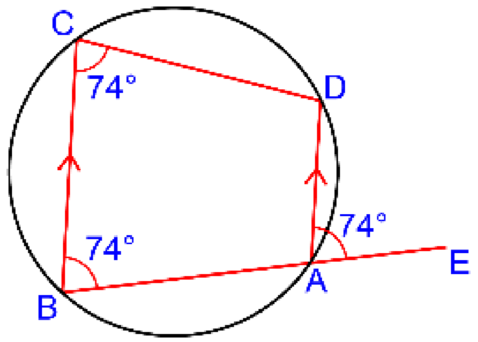
Angles in a cyclic quadrilateral

ABCD is a cyclic quadrilateral and BAE is a straight-line segment.
BC is parallel to AD.
What is the size of angle CBA?



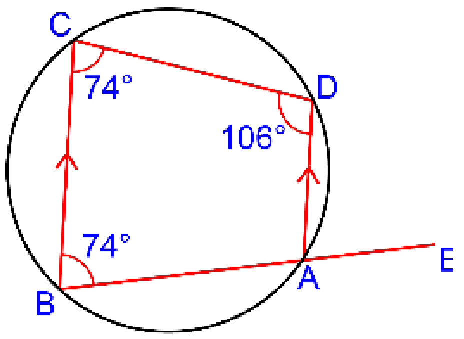
Method 1

Angle DAE = angle BCD = 74°
(exterior angle of a cyclic quadrilateral = interior opposite angle)
Angle CBA = angle DAE = 74° (corresponding)



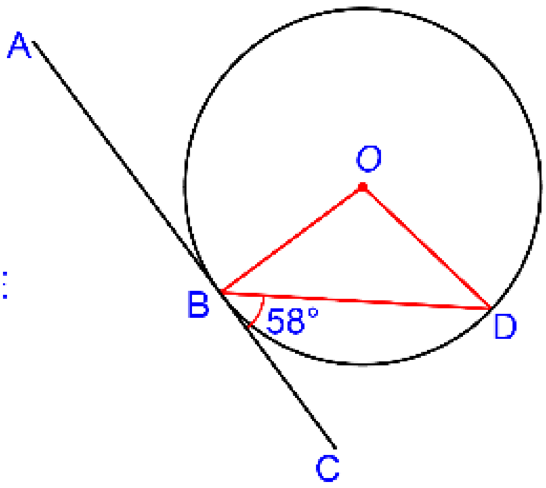
Method 2

Angle CDA = 180° − 74° = 106° (co-interior angles add to 180)
Angle CBA = 180° − 106° = 74° (opposite angles of a cyclic quadrilateral)



Combining circle theorems

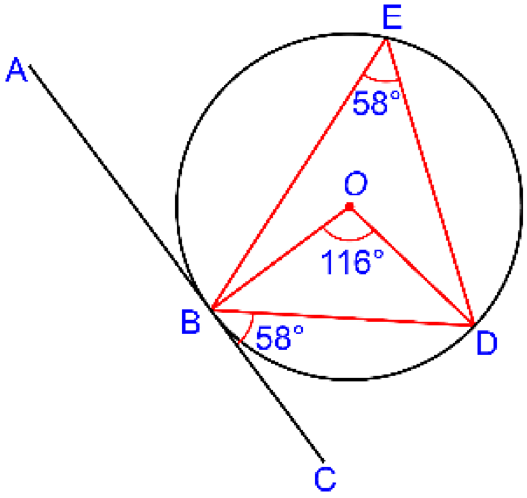
ABC is a tangent to the circle centre O. The tangent touches the circle at the point B. The angle DBC is 58°. What is the size of angle BOD?



The angle between the tangent and chord is given, so it is likely that the angle in the alternate segment is needed. Draw in the lines BE and DE, where E is any point on the circumference of the circle on the opposite side of chord BD from the angle DBC.

Angle BED = 58° (angle in the alternate segment)

Angle BOD = 116° (the angle at the centre is twice the angle at the circumference)



M5.10

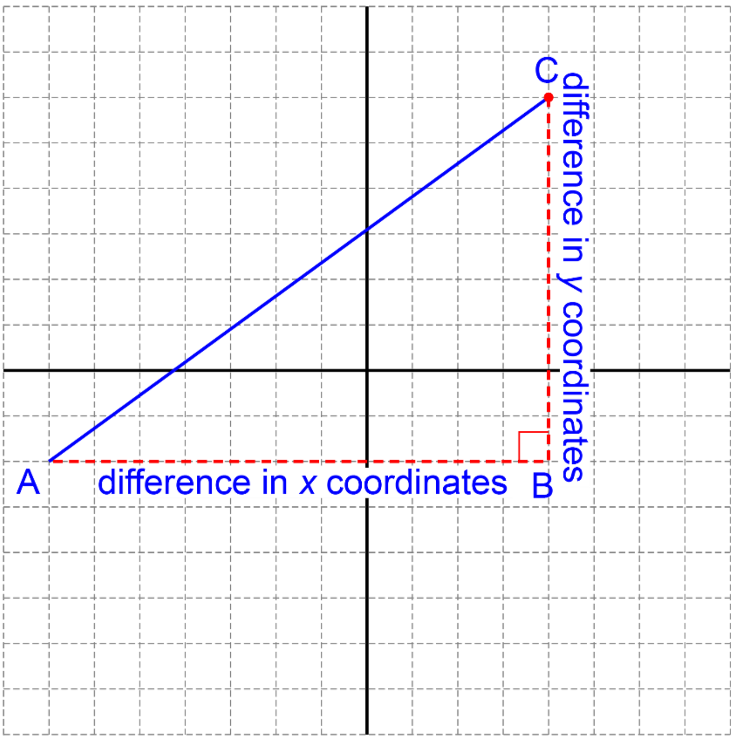
Solve geometrical problems on 2-dimensional coordinate axes.

Distance between two points

The distance between two points on coordinate axes can be found using Pythagoras' theorem.
If the points are A(a,b) and C(c,d), and B is the point where a vertical line from C meets a horizontal line from A then, using Pythagoras' theorem:

$AC^2 = AB^2 + BC^2$ or $AC^2 = (c - a)^2 + (d - b)^2 = (d - b)^2 + (c - a)^2$

(distance between two points)^2 = (difference in y coordinates)^2 + (difference in x coordinates)^2



Finding midpoints

The midpoint of two points A(a,b) and C(c,d) is $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

Problem solving

Geometric problems can be solved using coordinates.

Distance between two points

A is the point (2, −3) and B is the point (−4, 5).
What is the distance AB?

$AB^2 = (5 - -3)^2 + (-4 - 2)^2 = 8^2 + 6^2 = 64 + 36 = 100$
$AB = \sqrt{100} = 10$

Finding midpoints

A is the point (2, −3) and B is the point (−4, 5).
C is the midpoint of AB.

What are the coordinates of C?

The x-coordinate of C is $\frac{2+-4}{2} = -\frac{2}{2} = 1$
The y-coordinate of C is $\frac{-3+5}{2} = -\frac{2}{2} = 1$
The coordinates of C are (−1, 1).

Problem solving

ABCD is a trapezium with AB parallel to DC.

A is the point (−2, 4), B is the point (5, 4) and D is the point (−2, −3).

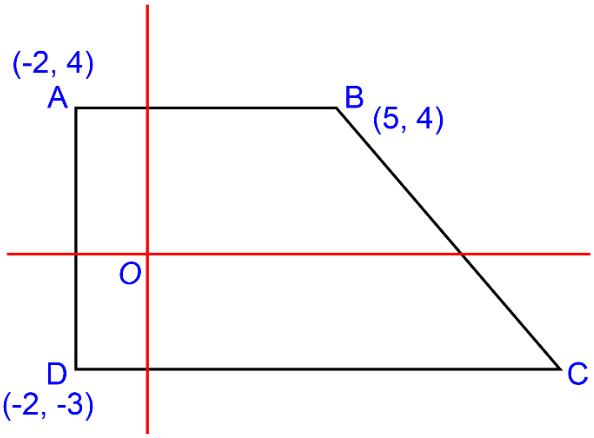
Angle BCD is an acute angle.

The length of BC is 12.

Find the length of DC.

Always draw a diagram. It does not need to be accurate, but it should show the important points.

Check the labelling of the diagram is in the correct order, and follow around the diagram either clockwise or anticlockwise consistently.



In this question it is important to

Show D on the same vertical axis as A

Show B on the same horizontal axis as A

Realize angle at D is 90

Draw BE parallel to AD

Realise that the acute angle at C (this is angle BCD)

means that DC is longer than AB

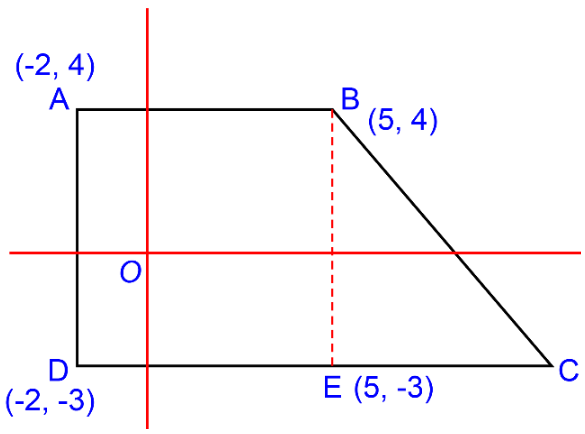
BE = AD = 4 − −3 = 7

BC = 12 (given)

$EC^2 = BC^2 - BE^2 = 12^2 - 7^2 = 144 - 49 = 95$

$EC = \sqrt{95}$

$DC = DE + \sqrt{95} = 7 + \sqrt{95}$



M5.11

Know the terminology *faces*, *surfaces*, *edges* and *vertices* when applied to cubes, cuboids, prisms, cylinders, pyramids, cones, spheres and hemispheres.

Note

These definitions are simply that: ideas defined by some individual, group or organisation. Different textbooks may define things in different ways so, for example, in some definitions a cylinder is a prism, in others it is not. In some books a face has to have straight-line edges, in others a face can be circular.

Examination questions state what is meant if there is any doubt, and they do not ask for definitions that are not universally agreed.

A **face** of a 3-dimensional figure is usually defined as a flat surface of a polyhedron, so a cube has 6 faces but a sphere has none. Most definitions require a face to be a polygon, so cylinders and hemispheres have no faces.

A **surface** of a 3-dimensional figure can refer to the entire outside of a 3-dimensional shape when talking about the surface area, so the surface area of a cube is 6 times the area of a face. Surface also refers to a curved surface, so a cone has a curved surface and a circular area. A cylinder has a curved surface and two circular areas.

An **edge** of a 3-dimensional figure is usually defined as a straight line joining two faces. Spheres, cylinders, cones and hemispheres have no edges as they have no straight lines or faces. A cube has 12 edges. (Some definitions allow an edge to join surfaces.)

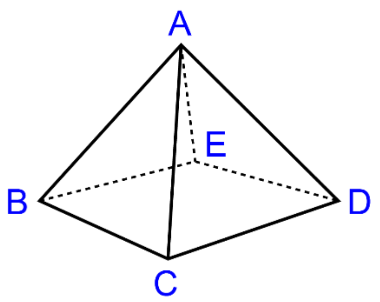
A **vertex** of a 3-dimensional figure is either a point where 2 or more edges of a polyhedron meet or the point of a cone.

The plural of vertex is vertices. A cylinder has no vertices. A cube has 8 vertices.

Faces, edges and vertices

How many a) faces b) vertices c) edges has a square-based pyramid?

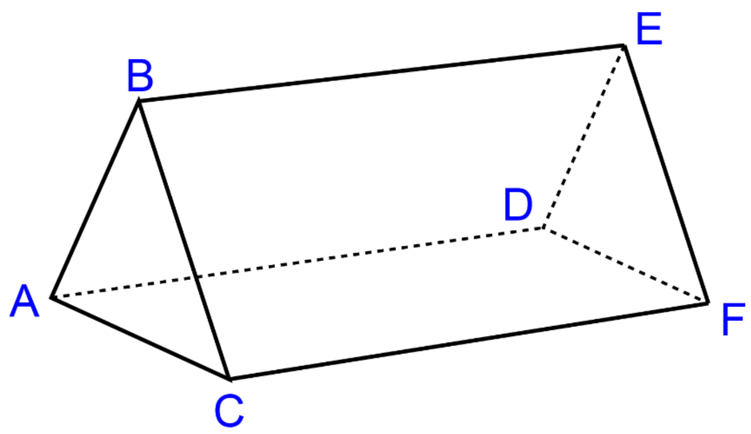
The faces are the triangles ABC, ACD, AED and ABE and the square BCDE, so there are 5 faces.
The vertices are the points A, B, C, D and E, so there are 5 vertices.
The edges are the lines AB, AC, AD, AE, BC, CD, DE, and EB, so there are 8 edges.



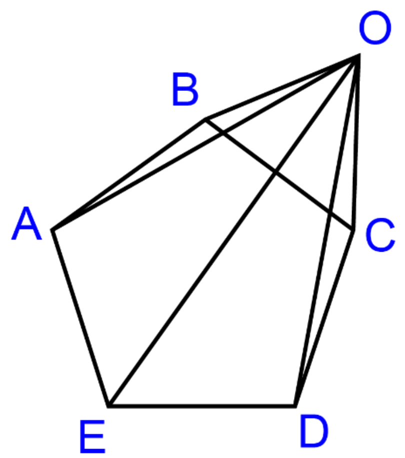
Faces, edges and vertices

Name 2 solids with 6 vertices.
Name a solid with 6 vertices and 5 faces.

Triangular prism and pentagonal pyramid
Triangular prism
The triangular prism has 6 vertices: A, B, C, D, E and F.
It also has 5 faces.



The pentagonal pyramid has 6 vertices: O, A, B, C, D and E.

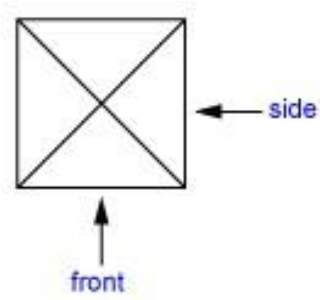


M5.12

Interpret plans and elevations of 3-dimensional shapes.

The **plan** of a 3-dimensional shape is the view from above looking down onto the object.

The plan of a square based right pyramid would look like this:



When you are drawing a plan and front and side elevations, you normally label the front and side onto the plan.

The plan gives us some idea of what the shape looks like, but tells us nothing about its height or whether or not it is standing on another shape.

The **front elevation** is what you would see if you were standing in front of the object looking at it directly. You are normally told which is the front.

The **side elevation** is what you would see if you were looking directly at the side of the object.

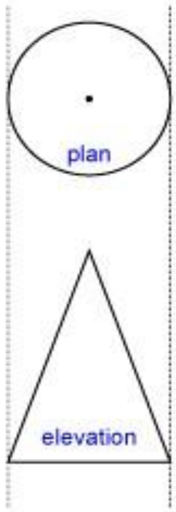
The front and side elevations of the square based pyramid are the same and are triangles with base the length of the base of the pyramid and height equal to the vertical height of the pyramid.

Drawing the plan and elevation of a solid

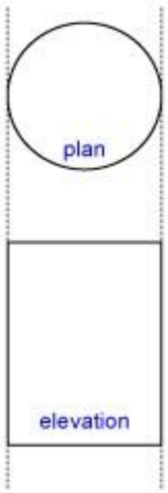
Draw the plan and elevation of

a cone
a cylinder

The plan of both shapes will be a circle, but the plan of a cone will have a central dot to show the vertex. The side elevation of the cone will be the same whichever side you look at it from, and is a triangle whose base length is the same as the diameter of the base of the cone and whose height is the same as the vertical height of the cone.
The side elevation of the cylinder will be the same whichever side you look at it from, and is a rectangle whose base length is the same as the diameter of the base of the cone. a)



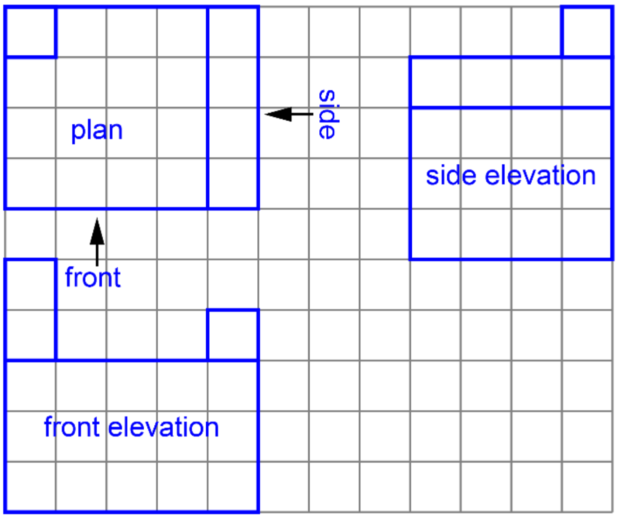
b)



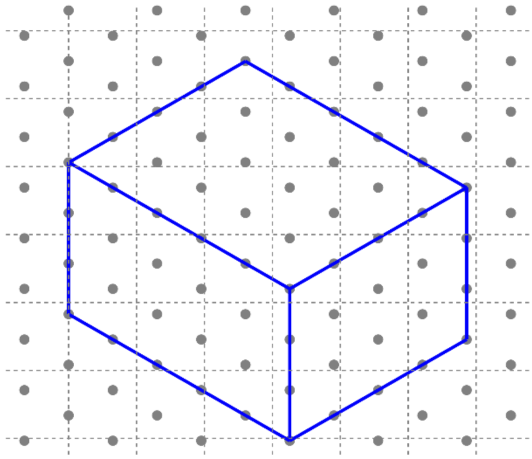
The dotted lines are not needed as part of the answer. Note that the side elevations do not show the bases' curved lines.

Drawing a solid from the plan and elevation

The plan, front elevation and side elevation of a solid are shown. Draw a possible solid, on isometric paper, which could have this plan and elevations.

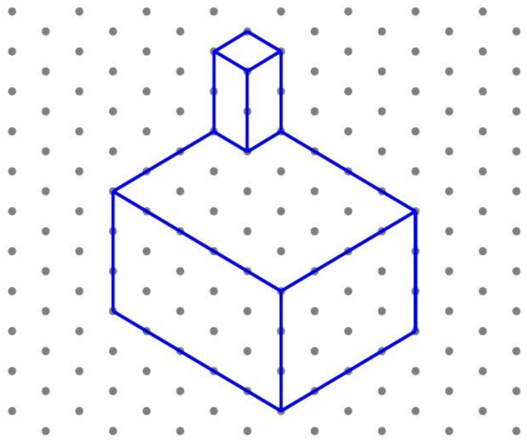


The plan shows a shape that is a rectangle 5 squares by 4 squares with two smaller rectangles.
The front and side elevations show that the shape could have a rectangular base, 3 squares high. More elevations would be needed to be certain, but the question only requires a solid which could have the plan and elevations given.

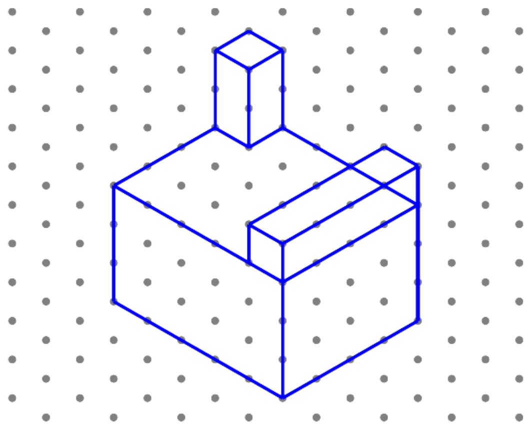


The plan shows two additional shapes on top of the base. One is a single square in cross section, and the front elevation shows it to be 2 cubes high.

The plan shows it to be placed on the back, right-hand corner of the base.



Finally, the plan shows a rectangular 4×1 block on the side edge, and the elevations both show this block to be one cube high, giving the final shape.



M5.13

Use and interpret maps and scale drawings.

Understand and use three-figure bearings.

Maps and scale drawings

A scale drawing is an enlargement of the original drawing, usually with a fractional scale factor. If a circle on the original drawing has radius 2 m, and the same circle on the scale drawing has a radius of 10 cm, then the scale factor of the enlargement is $\frac{2\text{ m}}{10\text{ cm}} = \frac{200\text{ cm}}{10\text{ cm}} = \frac{20}{1}$

This can be written as a ratio of 1 **:** 20 or simply as 10 cm represents 2 m.
A map is a form of scale drawing where you have a 2-dimensional drawing, or aerial view or plan, of a landscape.

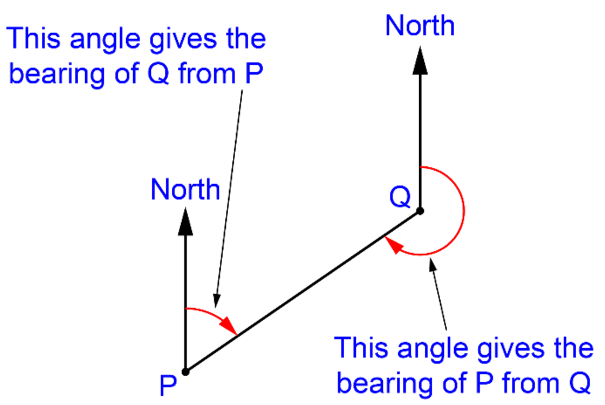
Bearings

Bearings are a way of writing the direction of one point from another. Bearings are always measured clockwise from north. They are written as a 3-digit number of degrees so a bearing of 50° from north would be written as 050°.

To measure the bearing of point P from point Q, draw a north line at the point the bearing is being measured from – in this case Q.

North lines are drawn straight up the page.

Join Q to P and measure the clockwise angle that this line makes with the north line at Q. Write this angle as a 3-figure bearing.
To measure the bearing of point Q from point P, draw a north line at the point the bearing is being measured from – in this case P. Join P to Q and measure the clockwise angle that this line makes with the north line at P. Write this angle as a 3-figure bearing.



Problems involving scale drawings and bearings can be solved using scale drawing, geometry or trigonometry.

Maps and scale drawings

A map shows the villages of Riverside and Hilltop.
The distance between the village shop in Riverside and the post office in Hilltop is 2 km and the corresponding distance on the map is 8 cm.

Calculate the scale of the map giving your answer in the form 1 **:** n

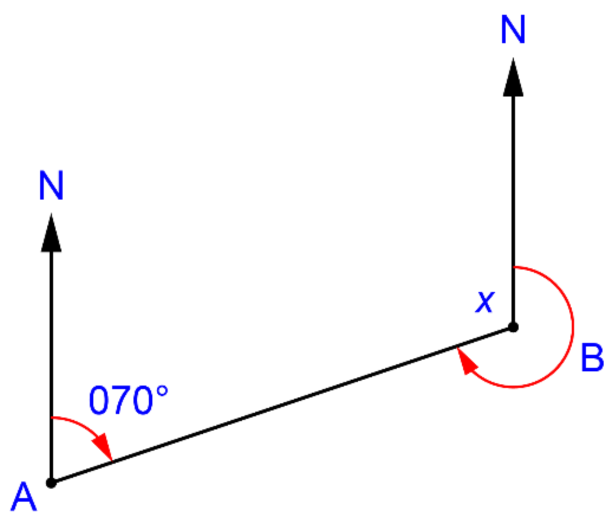
The actual distance is 2 km which is $2 \times 100 \times 1000 = 200\,000$ cm
The scale is 8 **:** 200 000 = (8 ÷ 8) **:** (200 000 ÷ 8) = 1 **:** 25 000

Bearings

The bearing of point B from point A is 070°. What is the bearing of point B from point A?

Always draw a diagram

Draw a point A on the page and draw a north line at A.
Draw an angle of 70° clockwise from the north line at A and extend the line.
Mark B as a point on the line. The distance has not been given and is not needed as the question is about angles.
The angle required is the angle between the north line at B and the line BA, measured clockwise.



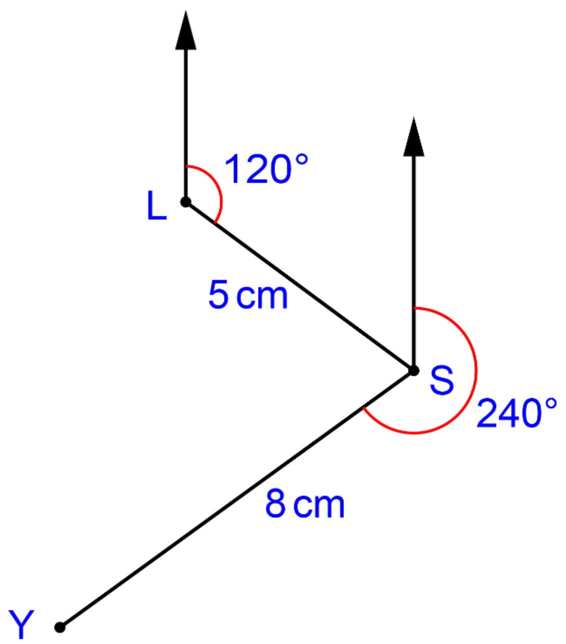
To find this angle, first find the angle marked x: $x + 70 = 180$ as the two north lines are parallel $x + 70 - 70 = 180 - 70 = 110°$

The angle required is $360 - x = 360 - 110 = 250°$

Bearings and geometry

A ship, S, is 10 km from a lighthouse, L, and the bearing of S from L is 120°.
A yacht, Y , is 10 km from S and the bearing of Y from S is 250°.
Calculate the bearing of Y from L.

The bearing from S is 250° and the two distances, S to L and S to Y, are the same, which will give either an isosceles triangle or an equilateral triangle.
Always draw a diagram, checking carefully that you are measuring angles clockwise from north.
To calculate the equal angles SLY and SY L, first calculate angle LSY and then use the properties of an isosceles triangle to calculate angles SLY and SYL.

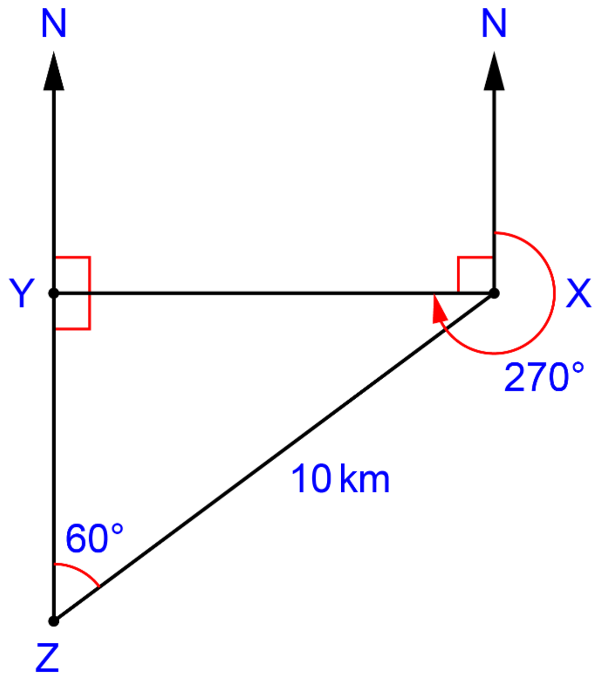


First find the angle marked x: x + 120 = 180 x = 60°
Angle LXY is 360 − 250 − 60 = 50°
Angle SLY = angle SYL = $\frac{180-50}{2}$= 65°
The bearing of Y from L is 120°+ angle SLY = 120 + 65 = 185°

Bearings and trigonometry

The point X is 10 km from Z on a bearing of 060° from Z.
Y is due north of Z.
Y is on a bearing of 270° from Y .
What is the exact value of the distance XY ?

Always draw a sketch diagram remembering to measure angles clockwise from north.
There is no need to measure angles and lengths accurately as the distance will be found by calculation.
Draw the north line at Z and draw an angle of approximately 60° clockwise and a point X on this line.
Mark XZ as 10 km. Draw a north line at X and put in the angle of 270° so Y is due east of X as well as being due north of Z. Complete the triangle XY Z.



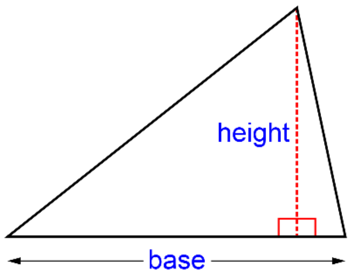
As angle ZYX is 90° then XY = 10 sin 60° = $10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$ km

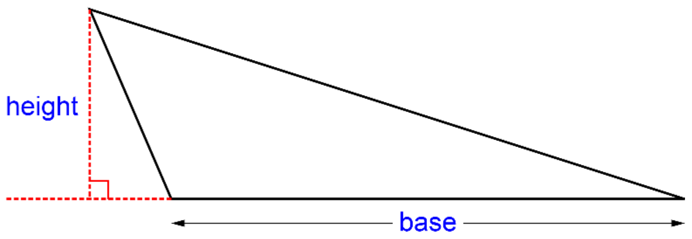
M5.14
Know and apply formulae to calculate:

a. the area of triangles, parallelograms, trapezia
b. the volume of cuboids and other right prisms.

Area of a triangle

The area of a triangle is $\frac{1}{2}$ (base × height)
The height must be measured perpendicular to the base.

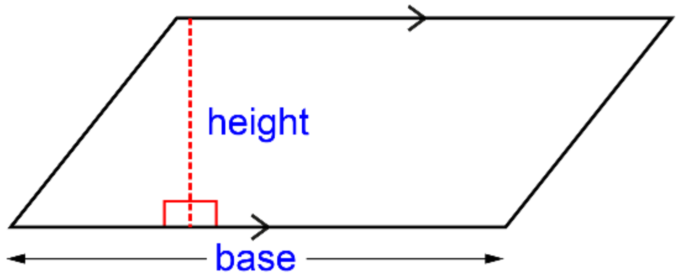




Area of a parallelogram

The area of a parallelogram is base × height

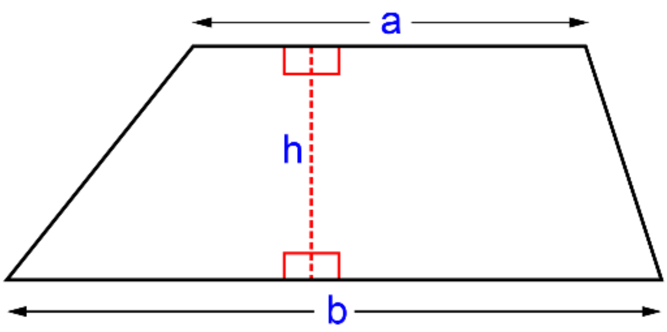
The height is the perpendicular distance between the base and the side parallel to the base.



Area of a trapezium

The area of a trapezium is $\frac{1}{2}$ × height × (the sum of the parallel sides) The height is the perpendicular distance between the parallel sides.

On the diagram: area = $\frac{1}{2}$ h (a + b)



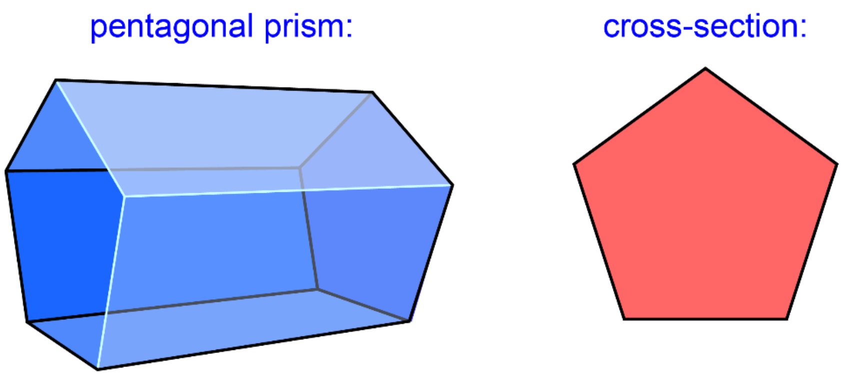
Volume of a right prism

The volume of a right prism is: area of cross-section × length

Definitions

A right prism is a polygon with parallel congruent ends and with rectangles joining those ends. The length of a right prism is the perpendicular distance between its ends.

A right prism has a constant cross-section when cut parallel to its ends, and that cross-section is a shape congruent to the shape of the ends.

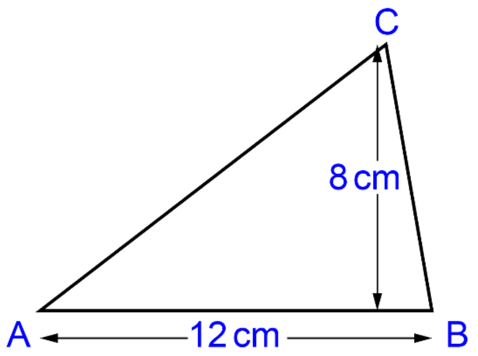


The right prism is usually named after the shape of its constant cross-section, for example triangular prism, hexagonal prism; but a right prism with a rectangular cross-section is called a cuboid or, if all its edges are the same length, a cube.

Area of a triangle

ABC is a triangle.

The length of side AB is 12 cm and the perpendicular distance of C from AB is 8 cm.
What is the area of triangle ABC?

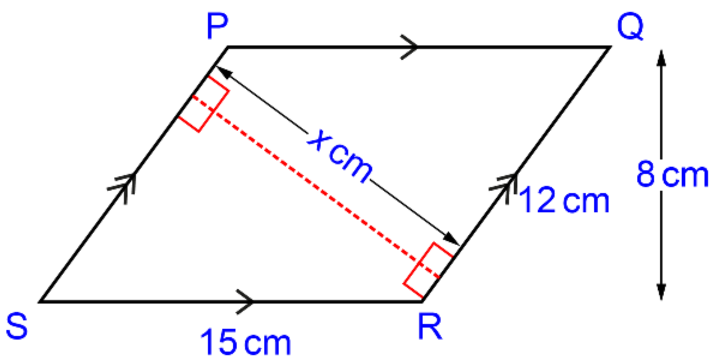


Area of triangle = $\frac{1}{2}$ base × perpendicular height = $\frac{1}{2}$ × 12 × 8 = 48 cm^2

Area of a parallelogram

PQRS is a parallelogram.
The side, SR, of the parallelogram PQRS is 15 cm.
The side QR of the parallelogram PQRS is 12 cm. The perpendicular distance between SR and PQ is 8 cm.
What is the area of PQRS?
What is the perpendicular distance between QR and PS?

Draw a diagram.

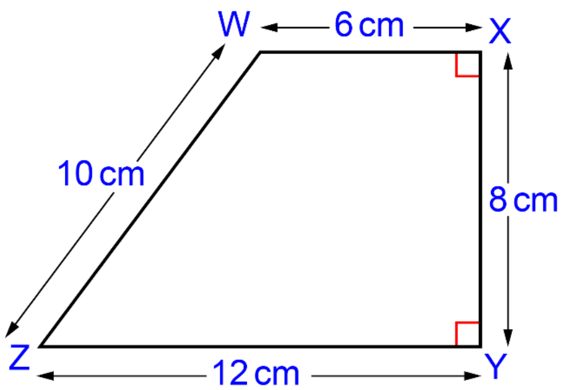


The area of PQRS, taking SR as the base, is 15 × 8 = 120 cm^2
The area of PQRS, taking QR as the base, is 12 × x = 120 cm^2
If 12x = 120 then 12x ÷ 12 = 120 ÷ 12 x = 10 cm

Area of a trapezium

WXY Z is a trapezium.
WX is parallel to ZY and XY is perpendicular to WX.
WX = 6 cm, XY = 8 cm, ZY = 12 cm and WZ = 10 cm What is the area of WXY Z?

Draw a diagram.



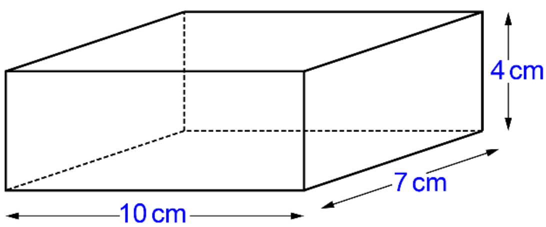
Area = $\frac{1}{2}$ × height × (the sum of the parallel sides)

Area = $\frac{1}{2} \times 8 \times (6 + 12) = 4 \times 18 = 72$ cm^2

The 10 is not used as it is not needed.

Volume of a cuboid

A rectangular block is a cuboid as shown in the diagram.

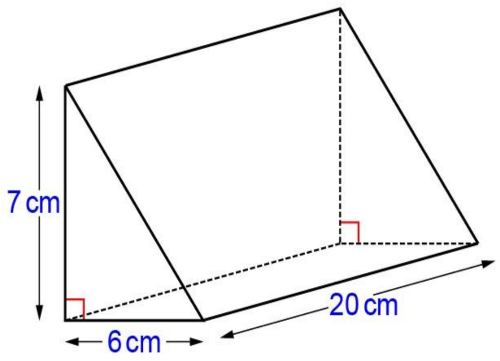


What is the volume of the block?

With a cuboid any face can be taken as the cross-section.
Area of the cross-section is 4 × 7 = 28 cm^2
Length = 10 cm
Volume = 28 × 10 = 280 cm^3

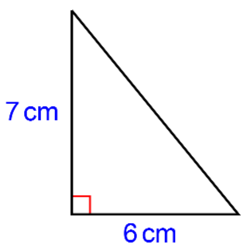
Volume of right prism

The cross-section of a right triangular prism is a right-angled triangle with shorter sides 6 cm and 7 cm. The length of the prism is 20 cm.



What is the volume of the prism?

The cross-section is the right-angled triangle shown.



The area of the triangle is $\frac{1}{2} \times 6 \times 7 = 21$ cm^2

The volume of the prism is area of cross-section × length = $21 \times 20 = 420$ cm^3

M5.15

Know the formulae:

a. circumference of a circle = $2\pi r = \pi d$
b. area of a circle = πr^2
c. volume of a right circular cylinder = $\pi r^2 h$

Formulae relating to spheres, pyramids and cones will be given if needed.

Use formulae to calculate:

a. perimeters of 2-dimensional shapes, including circles
b. areas of circles and composite shapes
c. surface area and volume of spheres, pyramids, cones and composite solids.

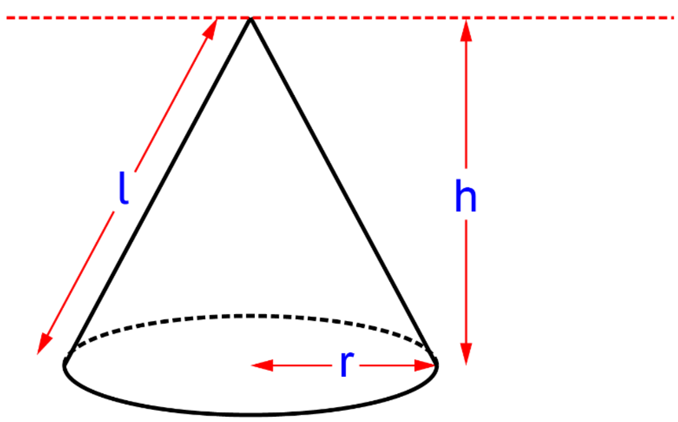
Formulae

The formulae which must be learned are:

- Circumference of a circle of radius r is $2\pi r$ (or πd where d is the diameter)
- Area of a circle of radius r is πr^2
- Volume of a right circular cylinder of radius r and height h is $\pi r^2 h$

The formulae which will be given are:

- Volume of a sphere of radius r is $\frac{4}{3}\pi r^3$
- Surface area of a sphere of radius r is $4\pi r^2$
- The volume of a cone of base radius r and perpendicular height h is $\frac{1}{3}\pi r^2 h$
- The surface of the curved area of a cone of base radius r and slant height l is $\pi r l$



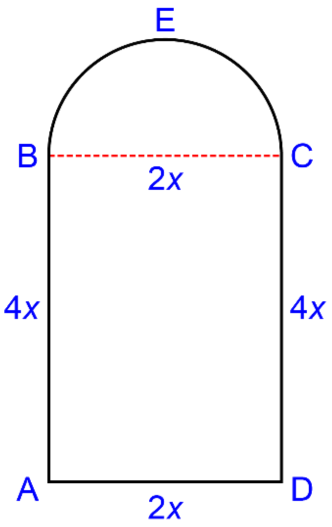
- The volume of a pyramid of base area A and perpendicular height h is $\frac{1}{3}Ah$

You can use these to find:

perimeters of simple and compound shapes in 2 dimensions including circles areas of circles and compound shapes surface areas and volumes of simple and compound shapes in 3 dimensions.

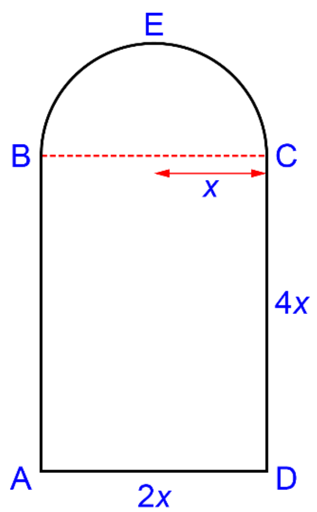
Perimeters of simple and compound shapes

A window is in the shape of a rectangle ABCD with a semicircle BEC drawn on the side BC as in the diagram.
The side BC of the rectangle is 2x cm and the side AB is 4x cm.
The perimeter of the window ABECD is 24(10 + π)



What is the length of the side CD?

The perimeter, P = AB + CD + AD+ the length of arc BEC
P = 4x + 4x + 2x+ the length of arc BEC (i)
The radius of the semicircle BEC = x
The length of the arc BEC = πx

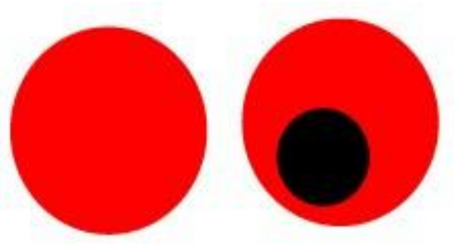


Substitute in (i)
P = 10x + πx = x(10 + π) = 24(10 + π) so x = 24
If x = 24 cm, then CD = 4x = 96 cm

Areas of circles and compound shapes

A circle of radius 4 cm is cut from a circle of radius 8 cm.
What fraction of the area of the circle of radius 8 cm remains?

Method 1



The area of the red circle is $\pi r^2 = 64\pi$
The area of the black circle is $\pi r^2 = 16\pi$
The red area remaining is $64\pi - 16\pi = 48\pi$
The fraction of the original red area is $\frac{48\pi}{64\pi} = \frac{3}{4}$

Method 2

The lengths on the black circle are $\frac{1}{2}$ the lengths of the red circle so the area of the black circle is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ of the area of the red circle so the difference is $\frac{3}{4}$ of the red area.

Surface areas and volumes

A model of an ice cream cone is a solid cone of base radius 10 cm and height 25 cm, attached to a solid hemisphere of radius 10 cm so that the circular flat surfaces of the hemisphere and cone meet exactly.

The volume of a cone is $\frac{1}{3}\pi r^2 h$ and the surface of the curved area is $\pi r l$.

The volume of a sphere is $\frac{4}{3}\pi r^3$ and the surface area is $4\pi r^2$.

What is:

the surface area

the volume of the model?

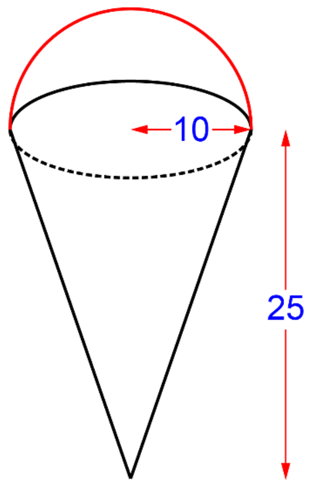
The surface area, S, of the model is $\pi r l + \frac{4\pi r^2}{2} = \pi r l + 2\pi r^2$

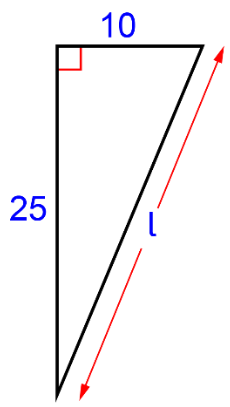
To find l use Pythagoras' theorem

$l^2 = 10^2 + 25^2 = 100 + 625 = 725$

$l = \sqrt{725} = 5\sqrt{29}$

$S = \pi \times 10 \times 5\sqrt{29} + 2 \times \pi \times 10^2 = 50(\sqrt{29} + 4)\pi$





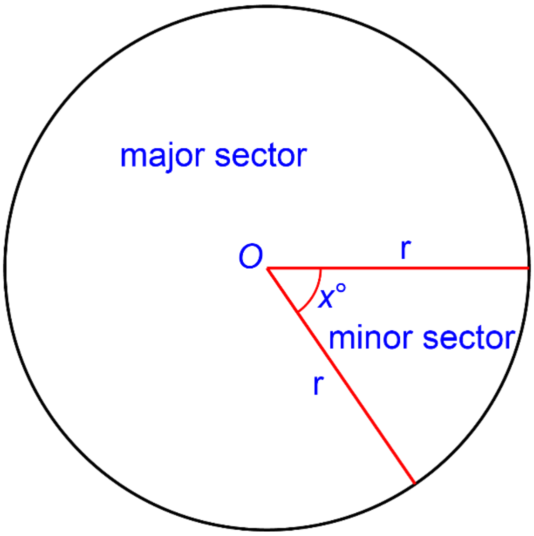
The volume, V , of the model is $\frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$

$V = \frac{1}{3} \times \pi \times 10^2 \times 25 + \frac{2}{3} \times \pi \times 10^3 = \frac{100\pi}{3}(25 + 2 \times 10) = \frac{4500\pi}{3}$

M5.16

Calculate arc lengths, angles and areas of sectors of circles.

A sector of a circle centre O is an area of the circle bounded by 2 radii and an arc of the circle.
2 radii divide a circle into 2 sectors, the major sector and the minor sector.



Sector angle

The angle between the 2 radii is called the sector angle.

Sector area

If the sector angle is x° then the area of the sector is $\frac{x}{360} \times \pi r^2$

Arc length

If the sector angle is x° then the length of the arc is $\frac{x}{360} \times 2\pi r$

Sector angle and area

A sector of a circle, C, has an angle of 70°. The radius of C is 12 cm.
What is the area of the sector, in cm^2?

Sector area = $\frac{x}{360} \times \pi r^2$ where x is the sector angle

Area = $\frac{70}{360} \times \pi \times 12^2 = 28\pi$ cm^2

Sector angle and arc length

A sector of a circle, C, has an angle of 80°. The radius of C is 15 cm.
What is the perimeter of this sector, in cm?

The length of the arc of the sector is $\frac{80}{360}$ × 2 × π × 15 cm = $\frac{20\pi}{3}$ cm

The perimeter, P, is 2r + the arc length

P = 2 × 15 cm + $\frac{20\pi}{3}$ cm = 10(3 + $\frac{2\pi}{3}$) cm

Arc length and sector angle

A sector of a circle with radius 18 cm has an arc length of 15π cm. What is the angle of this sector?

Let the sector angle be x°.

The arc length is $\frac{x}{360}$ × 2 × π × 18 cm = 15π cm Simplify the left-hand side of the equation:

$\frac{x}{360}$ × 2 × π × 18 cm= $\frac{x\pi}{10}$ cm

If $\frac{x\pi}{10}$ = 15π then

x = 150 and the sector angle is 150°

M5.17

Apply the concepts of congruence and similarity in simple figures, including the relationships between lengths, areas and volumes.

Congruent figures are identical in shape and size – all corresponding lengths and angles are equal. Similar figures are identical in shape, but one figure is an enlargement of the other. All corresponding angles are the same, and corresponding lengths are in the same ratio. All equilateral triangles are similar as are all spheres and all cubes.

Length, area and volume

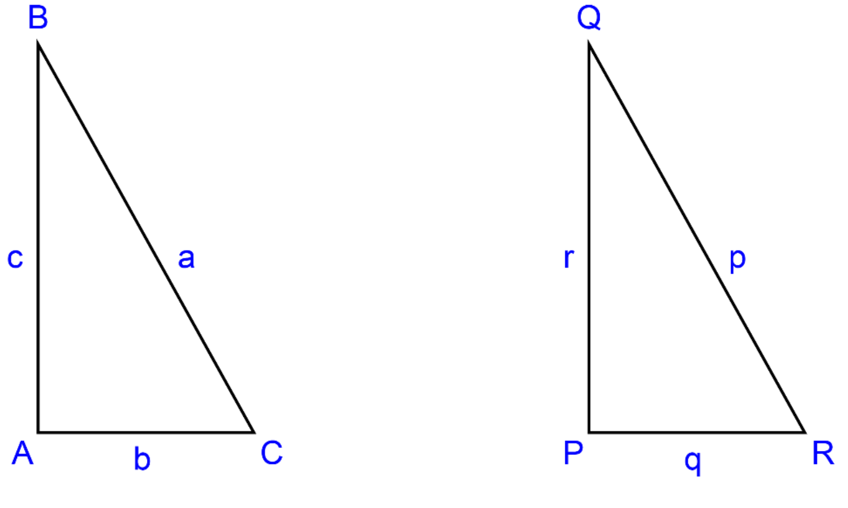
If a shape, X, is enlarged with scale factor n to give a similar figure, Y, then the area of Y is n^2 times the area of X and the volume of Y is n^3 times the volume of X.

Congruent figures

ABC and PQR are two right-angled triangles.
The hypotenuse BC of triangle ABC and the hypotenuse of triangle QR of triangle PQR are both 10 cm.
Is the statement

'Triangle ABC is congruent to triangle PQR' sometimes true, always true, or never true?

Draw a diagram.



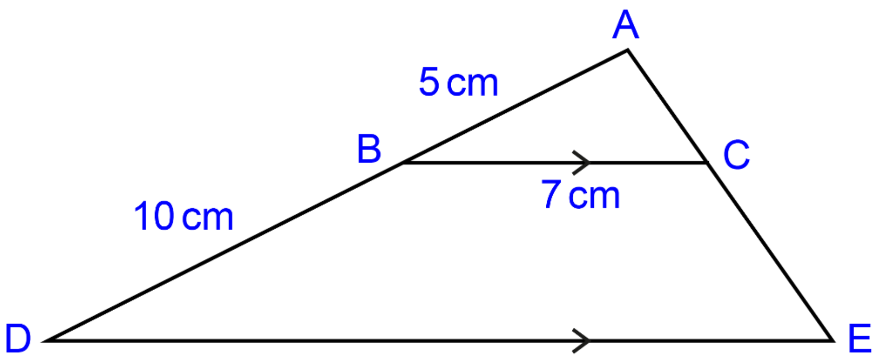
By Pythagoras' theorem: $b^2 + c^2 = 10^2 = q^2 + r^2$

This is true if b = q and c = r or if b = r and c = q and in these cases the triangles are congruent.
It is also true if $b^2 = 64$ and $c^2 = 36$ and $q^2 = r^2 = 50$ in which case the triangles are not congruent.

This means that the statement is sometimes true.

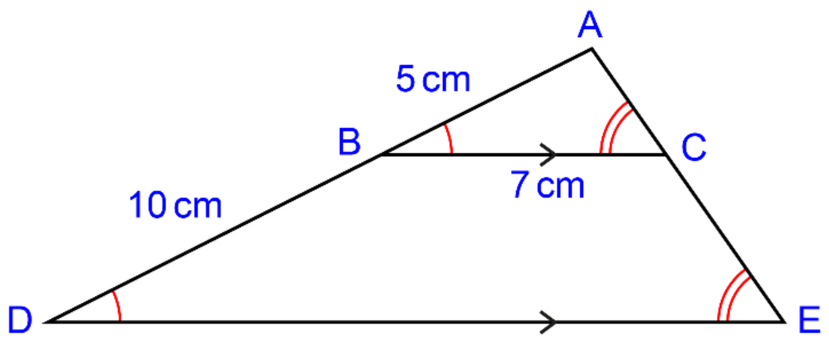
Similar figures

ADE is a triangle.
B is a point on AD and C is a point on AE.
BC is parallel to DE.
AB = 5 cm, BD = 10 cm and BC = 7 cm



Show that triangle ABC is similar to triangle ADE.
Find the length of DE.

The triangles have angle A in common
Angle ABC = angle ADE (corresponding angles)
Angle ACB = angle AED (corresponding angles)
So, triangle ABC is similar to triangle ADE as they have the same angles.



As the triangles are similar then corresponding sides are in the same ratio.
AB : AD is 5 : 15 or 1 : 3
BC : DE is also 1 : 3 as the triangles are similar so DE = 3 × 7 = 21 cm

Length, area and volume

The mass of the ingredients in an individual chocolate bar is 25 g and the wrapper used to cover the chocolate bar has an area of 60 cm^2.
The length, width and height of the ‘family bar’ of the same chocolate are twice the length, width and height of the individual bar and wrapper.
What is the area of the wrapper of the family bar?
What is the mass of the ingredients in the family bar?

The lengths are all multiplied by 2, so corresponding areas are multiplied by 2^2.
The area of the wrapper of the family bar is 60 cm × 2^2= 240 cm^2.
The lengths are all multiplied by 2 so corresponding volumes are multiplied by 2^3.
The mass of the ingredients of the chocolate bar is proportional to the volume so the mass of the ingredients of the family bar is 25 g × 2^3 = 200 g

M5.18

Know and use the trigonometric ratios:

$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

Apply these to find angles and lengths in right-angled triangles and, where possible, general triangles in 2- and 3dimensional figures.

Know the exact values of $\sin\theta$ and $\cos\theta$ for θ = 0°, 30°, 45°, 60°, 90°.

Know the exact values of $\tan\theta$ for θ = 0°, 30°, 45°, 60°.

Candidates are not expected to recall or use the sine or cosine rules.

Note

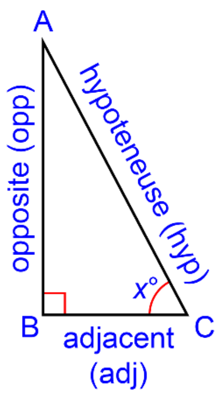
You need to know the exact values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for θ = 0°, 30°, 45° or 60° and of $\sin 90°$ and $\cos 90°$ either by learning them or learning how to derive them.

Sine, cosine and tangent

In a right-angled triangle ABC:

the side AB is opposite the angle BCA the side AC is the longest side of the right-angled triangle and is opposite the right angle. It is called the the side BC is adjacent to (beside) the angle BCA and the right angle.

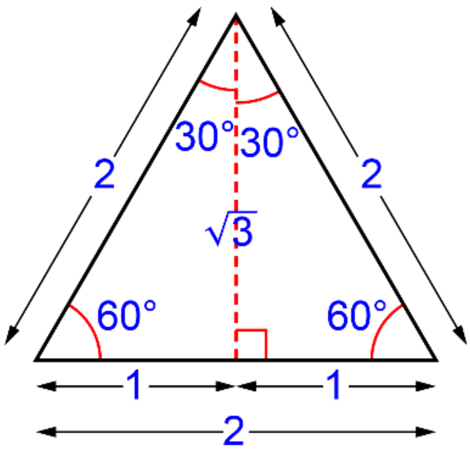
If the angle BCA is x° then these are three of the trigonometric ratios.



$\sin x° = \frac{\text{opposite}}{\text{hypotenuse}}$ $\cos x° = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\tan x° = \frac{\text{opposite}}{\text{adjacent}}$

Trigonometric ratios for 30° and 60°

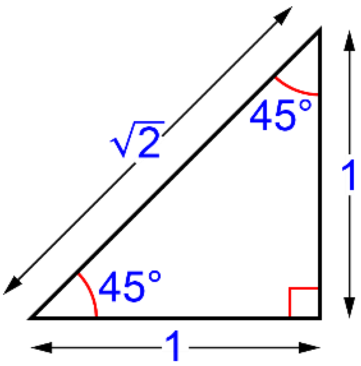
You can derive the values of the trigonometric ratios for angles of 30° and 60° from the following equilateral triangle of side 2 units, in which the perpendicular height ( ) of the triangle has been determined by Pythagoras' theorem:



Or you can learn the values: sin 30° = cos 60° = $\frac{1}{2}$; sin 60° = cos 30° = $\frac{\sqrt{3}}{2}$; tan 30° = $\frac{1}{\sqrt{3}}$; tan 60° = $\sqrt{3}$

Trigonometric ratios for 45°

You can derive the values of the trigonometric ratios for angles of 45° from the following isosceles right-angled triangle, in which the equal sides are of length 1 unit and the hypotenuse has been calculated using Pythagoras' theorem:



Or you can learn that sin 45° = cos 45° = $\frac{1}{\sqrt{2}}$ and tan 45° = 1

Problems in 2 dimensions

You can use trigonometric ratios to solve problems in 2 dimensions.

Problems in 3 dimensions

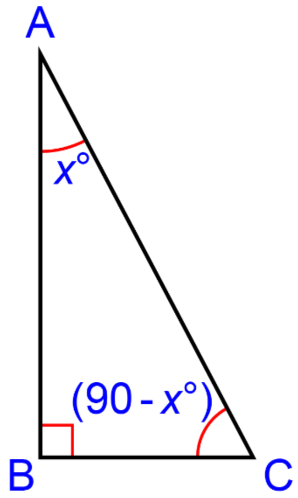
You can use trigonometric ratios to solve problems in 3 dimensions.

Trigonometric ratios for 0° and 90°

sin 90° = cos 0° = 1 and sin 0° = cos 90° = 0

Sine, cosine and tangent

In the right angled triangle ABC, angle ABC = 90° and angle BAC = x°
Find an expression for cos x°



$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{AC}$

Note: This is the same as calculating the sine of the angle at C, i.e.

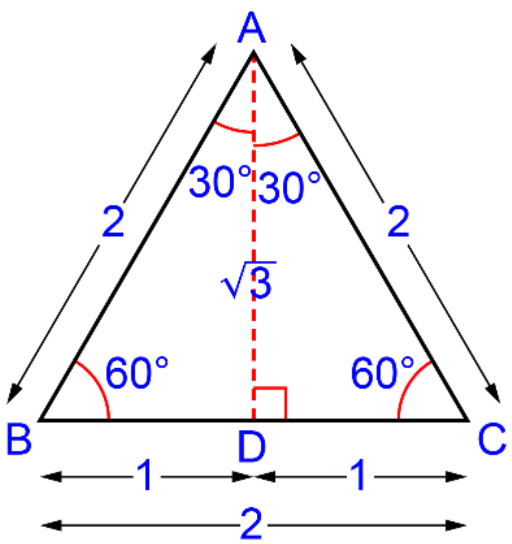
$\sin 90 - x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AB}{AC}$

Trigonometric ratios for 30° and 60°

Show that sin 60° = $\frac{\sqrt{3}}{2}$

ABC is an equilateral triangle of side 2.
D is a point on BC such that BD = DC = 1
AD is a line of symmetry of the triangle so:
Angle BAD = angle DAC = 30° and angle BDA = angle ADC = 90°
Using Pythagoras' theorem: $AD^2 = AB^2 – BD^2 = 4 – 1 = 3$
So AD = $\sqrt{3}$

$$\sin 60° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$$



Note: You do not have to show this reasoning each time you use the trigonometric ratios for 30° and 60°, but it is worth drawing a quick sketch if you want to check.

Trigonometric ratios for 45°

Show that sin 45° = $\frac{1}{\sqrt{2}}$

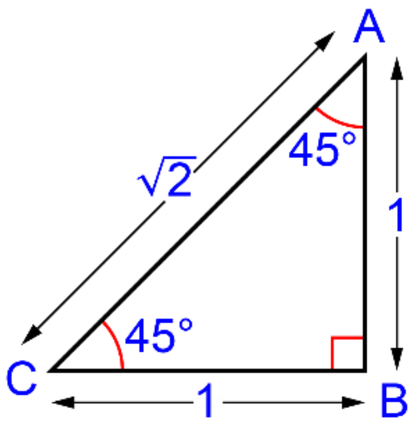
ABC is an isosceles triangle with angle ABC = 90° and AB = BC = 1

As the triangle is isosceles, angle BAC = angle BCA = $\frac{180-90}{2}$ = 45°

By Pythagoras' theorem, $AC^2 = AB^2 + BC^2 = 1 + 1 = 2$

So AC = $\sqrt{2}$

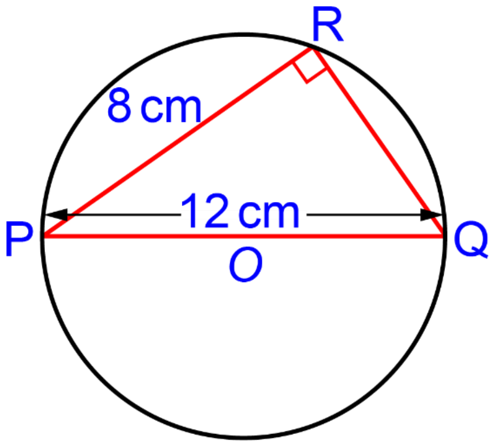
sin 45° = sin BAC = $\frac{BC}{AC} = \frac{1}{\sqrt{2}}$



Problems in 2 dimensions

A circle has centre O and radius 6 cm.
PQ is the diameter of the circle, and R is a point on the circumference of the circle. PR is 8 cm.
Find the value of sin RPQ

Always draw a diagram.

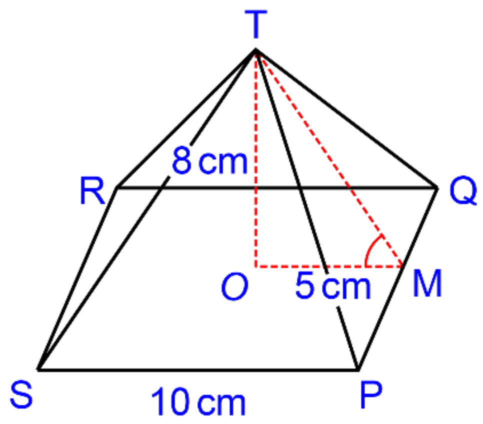


Angle PRQ = 90° angle in a semicircle
PQ = 2 × 6 cm = 12 cm twice the radius
sin RPQ = $\frac{RQ}{PQ}$
Using Pythagoras' theorem: $RQ^2 = PQ^2 - PR^2 = 144 - 64 = 80$
So RQ = $\sqrt{80} = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}$
sin RPQ = $\frac{RQ}{PQ} = \frac{4\sqrt{5}}{12} = \frac{\sqrt{5}}{3}$

Problems in 3 dimensions

PQRST is a square-based pyramid with square base PQRS of side 10 cm and vertical height 8 cm. What is the sine of the angle that the side PTQ makes with the base PQRS?

Always draw a diagram.
Let O be the centre of the square PQRS.
Let M be the midpoint of PQ.



OM = $\frac{1}{2}$SP = 5 cm

The angle that the side PTQ makes with the base PQRS is angle TMO.

sin TMO = $\frac{TO}{TM}$

By Pythagoras' theorem, $TM^2 = TO^2 + OM^2 = 8^2 + 5^2 = 64 + 25 = 89$

So TM = $\sqrt{89}$ cm

sin TMO = $\frac{TO}{TM} = \frac{8}{\sqrt{89}}$

M5.19

Apply addition and subtraction of vectors, multiplication of vectors by a scalar, and diagrammatic and column representations of vectors.

Use vectors to construct geometric arguments and proofs.

A vector is a way of describing how to move from one point to another. Vectors describe both direction and size.

Note:
Vectors are usually shown by bold letters, for example **b**.
b is written $\underline{b}$ in handwriting as underlining is the printing proof symbol for bold.
Vectors can be shown as column vectors or diagrammatically.
Adding and subtracting column vectors
Column vectors can be added or subtracted by adding or subtracting across the rows of the vectors.

$\binom{a}{b} + \binom{d}{c} - \binom{f}{e} = \binom{a+d-f}{b+c-e}$

Multiplying column vectors by a scalar

Column vectors can be multiplied by a scalar. A scalar is a quantity, which has magnitude only and not direction.

Velocity is a vector, speed is a scalar.

If $\mathbf{x} = \binom{a}{b}$ then $6\mathbf{x} = \binom{6a}{6b}$ and $-\frac{1}{2}\mathbf{x} = \begin{pmatrix} -\frac{1}{2}a \\ -\frac{1}{2}b \end{pmatrix}$

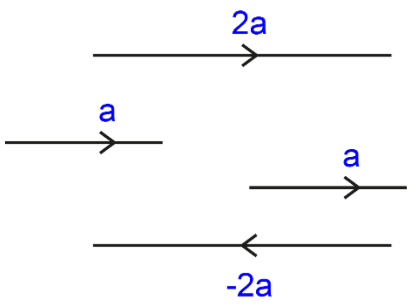
Parallel vectors

If two vectors **x** and **y** are parallel then **x = cy** where c is a constant.

Diagrammatic representation of vectors

If the vector **a** is represented diagrammatically by the line segment OA in the direction from O to A, then all line segments parallel to OA in the same direction and of the same length also represent the vector **a**. If a line segment is parallel to OA in the same direction but twice as long, it would represent the vector 2**a**.

If a line segment is parallel to OA in the opposite direction but twice as long, it would represent the vector −2**a**.

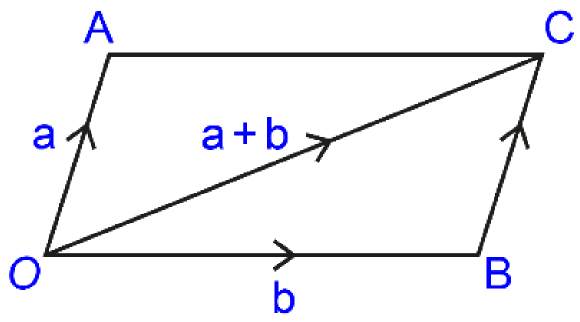


Triangular addition and subtraction of vectors

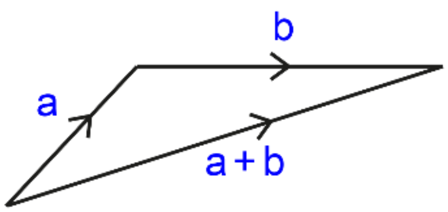
Vectors can be added and subtracted diagrammatically using triangles.

Example

For example, if OACB is a parallelogram, OA = **a** and OB = **b**
The vector OC represents the journey from O to C.
The alternative route from O to C is to travel along OB (**b**) and then BC.
BC is parallel to OA, equal in length to it, and in the same direction so BC = **a**
The journey from O to C in vector terms has the same effect as travelling from O to B and then B to C as both journeys start at O and end at C.
In vector terms: OC = OB + BC = **b** + **a** = **a** + **b**



Similarly, AB = AO + OB = −**a** + **b** = **b** – **a** and BA = BO + OA = −**b** + **a** = **a** – **b**
To add or subtract two vectors **a** and **b**, either
draw a parallelogram (as in the diagram above) with **a** and **b** as sides to construct the triangles needed to find **a** + **b** and **a** – **b**
or to add the two vectors, draw a triangle with **a** and **b** in the same sense, and join the start and end points.



Geometric proofs using vectors

Vectors can be used to prove geometric properties.

Adding and subtracting column vectors

Given **a** = $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **b** = $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ **c** = $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$ find **a** − **b** + **c**

a − **b** + **c** = $\begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 2--3+1 \\ -3-5+6 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

Multiplying column vectors by a scalar Given

a = $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **b** = $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ **c** = $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$ find 2**a** − 3**b**

If **a** = $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ then 2**a** = $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$

If **b** = $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$ then 3**b** = $\begin{pmatrix} -9 \\ 15 \end{pmatrix}$

2**a** − 3**b** = $\begin{pmatrix} 4 \\ -6 \end{pmatrix} - \begin{pmatrix} -9 \\ 15 \end{pmatrix} = \begin{pmatrix} 4--9 \\ -6-15 \end{pmatrix} = \begin{pmatrix} 13 \\ -21 \end{pmatrix}$

Multiplying column vectors by a scalar

If **x** = $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, **y** = $\begin{pmatrix} -6 \\ -1 \end{pmatrix}$, which of the following vectors is/are parallel to **x** − **y**?

$\begin{pmatrix} 16 \\ -4 \end{pmatrix}$

$\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 8 \\ 2 \end{pmatrix}$

x − **y** = $\begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} -6 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

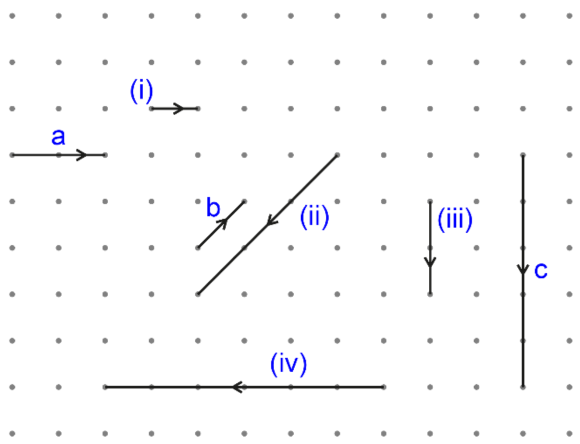
$\begin{pmatrix} 16 \\ -4 \end{pmatrix} = 2 \times \begin{pmatrix} 8 \\ -2 \end{pmatrix}$ so they are parallel

$\begin{pmatrix} -4 \\ 1 \end{pmatrix} = -\frac{1}{2} \times \begin{pmatrix} 8 \\ -2 \end{pmatrix}$ so they are parallel

$\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ is not a multiple of $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$ so they are not parallel

Diagrammatic representation of vectors

Label all the lines in this diagram with the correct vectors using **a**, **b** and **c**.

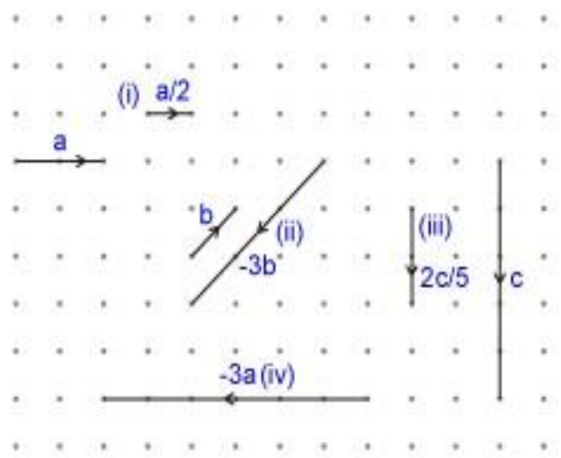


The line (i) is parallel to **a** and in the same direction but half the length, so it is $\frac{1}{2}$ **a**

The line (ii) is parallel to **b** in the opposite direction and 3 times as long, so it is −3**b**

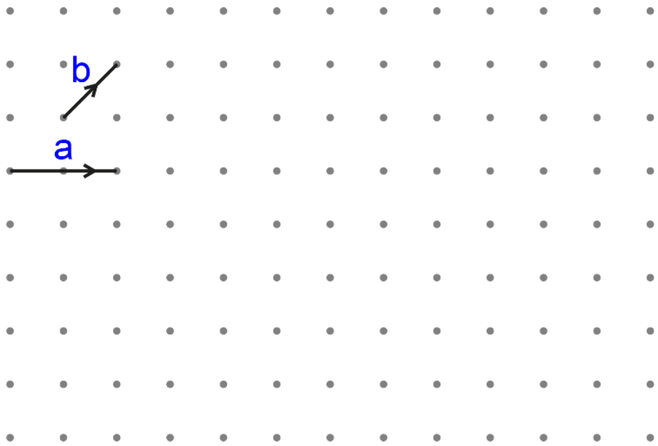
The line (iii) is parallel to **c** and in the same direction but $\frac{2}{5}$ the length, so it is $\frac{2}{5}$ **c**

The line (iv) is parallel to **a** in the opposite direction and 3 times as long, so it is −3**a**

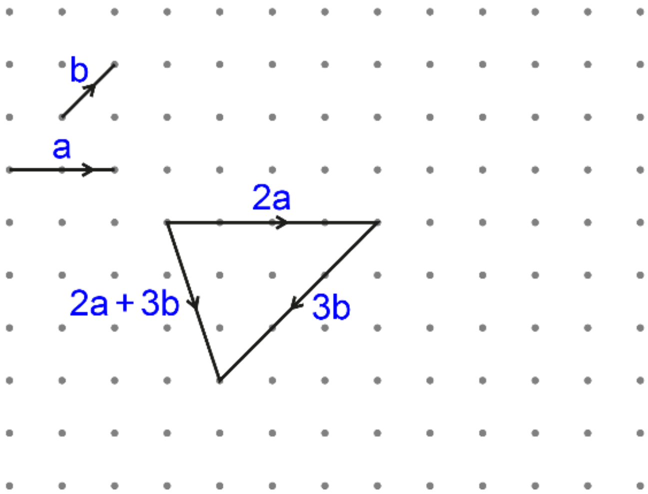


Triangular addition and subtraction of vectors

In this diagram, draw a vector equal to 2**a** + 3**b**

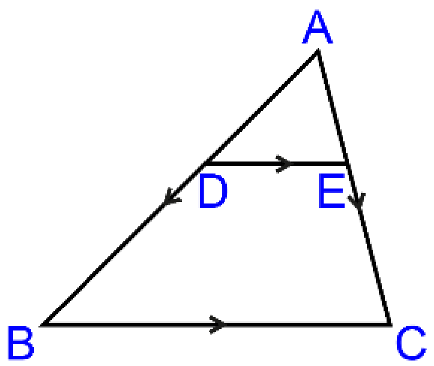


Draw the vector 2**a**, a line twice as long as **a**, parallel to it and in the same direction. Next draw 3**b** starting from the end of 2**a** and following on from **a**.



Geometric proofs using vectors

ABC is a triangle, D is the midpoint of AB, E is the midpoint of AC.
Show, using vectors, that DE is parallel to BC and is equal in length to half of it.



Draw a diagram.

Let AB = **x** and AC = **y**

BC = BA + AC = $-\mathbf{x} + \mathbf{y} = \mathbf{y} - \mathbf{x}$

AD = $\frac{1}{2}\mathbf{x}$ and AE = $\frac{1}{2}\mathbf{y}$

DE = DA + AE = $-\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} = \frac{1}{2}(\mathbf{y} - \mathbf{x}) = \frac{1}{2}$BC

As DE = $\frac{1}{2}$BC then DE is parallel to BC and equal in length to half of it.

M6. Statistics

M6.1

Interpret and construct tables, charts and diagrams, including:

a. two-way tables, frequency tables, bar charts, pie charts and pictograms for categorical data
b. vertical line charts for ungrouped discrete numerical data
c. tables and line graphs for time series data

Know the appropriate use of each of these representations.

Two-way tables for categorical data

A two-way table has both rows and columns forming cells. Each cell contains information relating to the row and column that it belongs to.

For example: 20 pupils were asked whether they preferred red or blue, and whether they preferred cats or dogs. If the results were put in the two-way table shown, you could say that 5 pupils preferred red to blue, and cats to dogs; 12 pupils preferred cats to dogs; and 9 pupils preferred red to blue.

	red	blue
cat	5	7
dog	4	4

Two-way tables can also include totals for the rows and columns. The overall total is where the row for total and the column for total cross.

	red	blue	Total
cat	5	7	12
dog	4	4	8
Total	9	11	20

Frequency tables for categorical data

A frequency table shows the number of times a particular event occurs. Frequency is often recorded using a tally chart and then the tallies are totalled to give the frequency.

Frequency tables are used to summarise data in an organised manner.

Bar charts for categorical data

A simple bar chart is a diagram in which the length of the bar is proportional to the frequency. A bar chart can be drawn with horizontal or vertical bars.

A grouped bar chart is used to show data split further by source. For example, sales figures by month of three shopworkers.

A stacked bar chart is used to show data split into further subcategories. For example, total sales of ice cream by month with each bar split into flavours of ice cream by frequency.

Bar charts are used to display data in an easy-to-read format and can be used to show subcategories.

Pie charts for categorical data

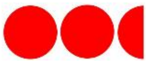
A pie chart is a diagram in which the angle of the sector is proportional to the frequency. Pie charts are usually circles. They can be constructed from raw data or from proportions. Sometimes pie charts are shown in 3 dimensions, but this is misleading as the angling of the pie chart and showing the depth mean that the sections represented at the front appear bigger than they are.

Pictograms for categorical data

A pictogram is a diagram in which a shape or symbol is used to represent a frequency. For example:

if

represents a frequency of 10, then



represents a frequency of 25.

A pictogram is used to display data in an easy-to-read diagram.

When drawing a pictogram, a key is needed to indicate what one symbol represents.

Vertical line charts for ungrouped discrete numerical data

These are similar to bar charts, but the bars have no width because the data is ungrouped. The bars are also separate as the data is discrete.

Vertical line graphs are used to display ungrouped, discrete numerical data.

Tables and line graphs for time series data

Time series data is data which has been recorded over a period of time, such as the average monthly temperature over a year. The data can be shown in a table, but is usually easier to interpret quickly if the data is shown as points on a graph. These points are usually joined by straight lines, which can be solid or dotted lines. A trend line can be drawn to show the pattern formed by the data more clearly.

Choosing an appropriate representation

Frequency tables and vertical line graphs are the easiest way of displaying categorical data if there are a large number of categories. Vertical lines enable easy comparison of frequencies but it is often easier to read the frequency accurately from a table.

For smaller numbers of categories, bar charts, pictograms and pie charts are a good way of showing frequencies visually. If the categories are further broken into subcategories then bar charts are usually appropriate.

Time series tables and line graphs are used when comparing data over time and for showing a trend.

Two-way tables

The 150 pupils in Year 10 study either French or Spanish at school but not both. The table shows some information about this. How many girls study Spanish?

	French	Spanish
boys	42	37
girls	44	

Let the number of girls studying Spanish be x.

The total number of pupils is 150 so 42 + 37 + 44 + x = 150 so x = 150 − 123 = 27

Frequency tables

80 students were asked to choose their favourite flavour of ice cream from those sold in the school shop. A tally chart is made of the results. Complete the frequency table.

flavour	tally	frequency
strawberry	卌 卌 卌 卌 II	
chocolate	卌 卌 卌 卌 卌 I	
vanilla	卌 卌 IIII	
banana	卌 卌 卌 III	

The tally chart shows vertical strokes to represent 1. A group of 5 is shown by 4 vertical strokes and a diagonal bar.

In the strawberry row, there are 4 groups of 5 and 2 singles, so the frequency is 20 + 2 = 22

In the chocolate row, there are 5 groups of 5 and 1 single, so the frequency is 25 + 1 = 26

In the vanilla row, there are 2 groups of 5 and 4 singles, so the frequency is 10 + 4 = 14

In the banana row, there are 3 groups of 5 and 3 singles, so the frequency is 15 + 3 = 18

flavour	tally	frequency
strawberry	卌 卌 卌 卌 II	22
chocolate	卌 卌 卌 卌 卌 I	26
vanilla	卌 卌 IIII	14
banana	卌 卌 卌 III	18

Check: 22 + 26 + 14 +18 = 80

Bar chart, simple and grouped

The 24 pupils in a class were asked to choose their favourite subjects from History, English and Art.

6 boys and 4 girls chose History, 3 boys and 5 girls chose English, and 3 boys and 3 girls chose Art.

Draw a bar chart to show the total number of pupils choosing each subject.

Draw a dual bar chart to show the number of boys and girls choosing each subject.

a) The total number of pupils choosing History is 10.

The total number of pupils choosing English is 8.

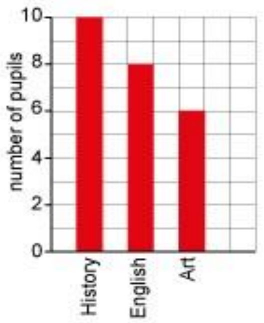
The total number of pupils choosing Art is 6.

The longest bar is of length 10 so the vertical axis is labelled from 0 to 10, and given the description 'number of pupils'.

A bar of length 10 is drawn for History, and the label History is put underneath the bar.

A bar of length 8 is drawn for English, and the label English is put underneath the bar.

A bar of length 6 is drawn for Art, and the label Art is put underneath the bar. Space is left between the bars for clarity. The bars could be drawn horizontally with the number of pupils labelled on the horizontal axis.



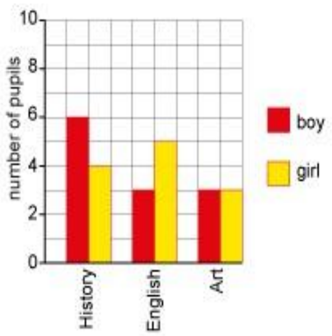
Bar chart showing the subject preferences of 24 pupils

b) 6 boys and 4 girls chose History.

The two bars are drawn side by side, with the label History beneath them.

The bars are colour coded for boys and girls, and a key is given stating which is which.

The process of drawing the bars is repeated for English and Art.

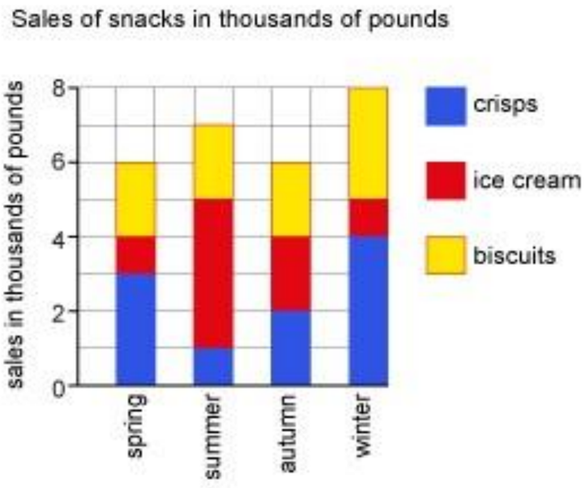


Bar chart showing the subject preferences of 24 pupils

Stacked bar chart

The stacked bar chart shown gives the total sales of 3 types of snacks in a shop during the 4 seasons of the year.

What is the difference, in thousands of pounds, between the sales of biscuits and crisps in the year?



Sales of biscuits: 2 + 2 + 2 + 3 = 9 thousand pounds

Sales of crisps: 3 + 1 + 2 + 4 = 10 thousand pounds

The difference is 10 − 9 = 1 thousand pounds

Pie charts

Students recorded the colour of 120 cars which passed the school gate during their lunch hour. 40% of the cars were silver.

12 cars were red. $\frac{1}{4}$ of the cars were blue.

The remainder of the cars were black.

Draw a pie chart to show this information, marking the size of the sector angles.

How many of the cars were black?

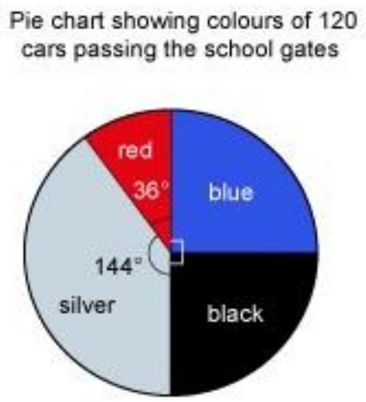
a) The object is to divide the 360° of the full circle into angles proportional to the number of cars that the sector represents.

40% of 360° is $\frac{40}{100} \times 360° = 144°$

12 cars is $\frac{12}{120} = \frac{1}{10}$ of the cars.

$\frac{1}{10}$ of 360° is $\frac{1}{10} \times 360° = 36°$

$\frac{1}{4}$ of 360° is $\frac{1}{4} \times 360° = 90°$



b) The black sector is $(360 - 144 - 36 - 90)° = 90°$

90° is $\frac{1}{4}$ of 360° so $\frac{1}{4}$ of the 120 cars are black. 30 cars are black.

The pie chart needs to be divided into sectors with angles of 144°, 36° and two sectors of 90°.

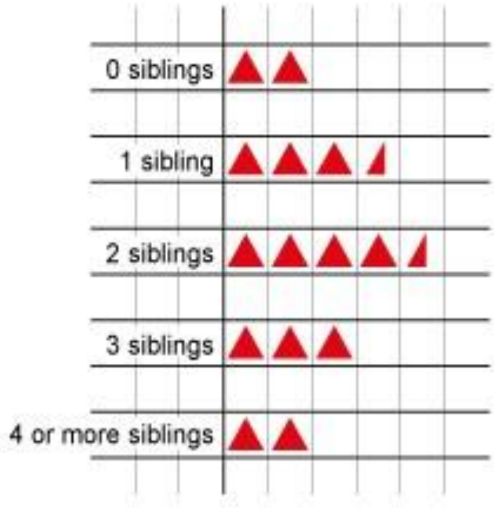
Label the sectors of the pie chart clearly and use a title for the chart. The angles do not have to be labelled as these can be measured.

Pictograms

60 students were asked how many siblings they have. The results are shown in the pictogram. The pictogram is missing its title and key.

How many students does each triangle represent?

How many students had 2 or more siblings?



a) There are a total of 2 + 3.5 + 4.5 + 3 + 2 = 15 triangles

15 triangles represent 60 students so 1 triangle represents 60 ÷ 15 = 4 students

b) Students having 2 or more siblings are represented by 4.5 + 3 + 2 = 9.5 triangles

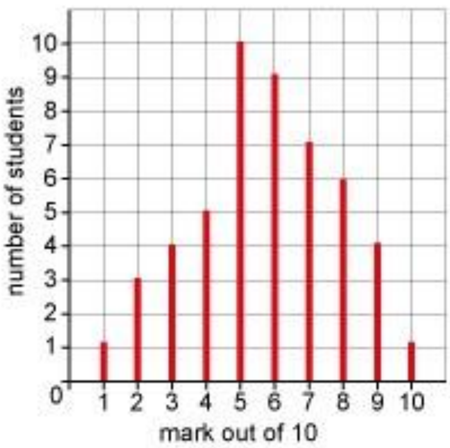
9.5 triangles represent 9.5 × 4 = 38 students

Vertical line charts

The line graph shows the marks out of 10 achieved by students in a test. The maximum mark in the test is 10, and no student scored 0 marks.

How many students took the test?

How many students achieved 6 marks or more in the test?



a) The total number of students is the total of all the lines:

1 + 3 + 4 + 5 + 10 + 9 + 7 + 6 + 4 + 1 = 50

b) The number of students who achieved 6 or more marks is the total length of the lines representing 6, 7, 8, 9 and 10 marks: 9 + 7 + 6 + 4 + 1 = 27

Tables and line graphs for time series data

The table shows the sales of ice cream in a shop for the 4 quarters of the years 2016, 2017 and 2018.

Sales of ice cream in thousands of pounds

	2016				2017				2018			
quarter	1	2	3	4	1	2	3	4	1	2	3	4
sales £1000	1.5	2	3.5	1.5	2	3	4	2	2.5	3.5	5	4

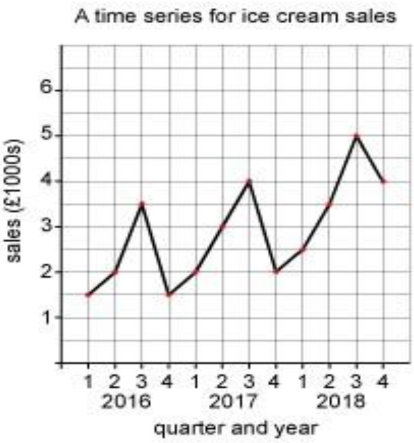
Draw the time series line graph for this data.

Comment on the trend in sales in the period 2016 to 2018.

In a time series graph, always put the time on the horizontal axis.

The scale on the horizontal axis must be linear (i.e. equal width intervals represent the same amount of time all along the axis) and continuous, and the axes should be labelled.

Plot the points and join with straight lines.



The graph shows a mixture of upward and downward trends, though overall sales increased from 2016 to 2018.

M6.2

Interpret and construct diagrams for grouped discrete data and continuous data:

a. histograms with equal and unequal class intervals
b. cumulative frequency graphs

Know the appropriate use of each of these diagrams.

Understand and use the term *frequency density*.

Discrete and continuous data

Discrete data is data which can only take certain fixed values.

For example, the number of pupils in a class is discrete as it has to be an integer and cannot be, for example, 23.3. The number of cars in a car park is discrete, and UK shoe sizes, e.g. 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$ etc. are discrete.

Continuous data is data which can take on any values in a range and not just particular values. For example, the heights of pupils in a class are not always whole numbers; a height could be 165.75 cm so the data set is continuous.

Class intervals

Inequalities are used to express class intervals.

Class intervals must be continuous so that the upper boundary of one class is the lower boundary of the next class.

When using class intervals, make sure that every number belongs to only one class interval.

Class intervals do not have to be of equal width.

If the heights of a group of students are measured and the results given to the nearest cm, then the data are continuous as the heights can take any value, even if they are later corrected for recording.

If the heights are grouped as:

150 cm to 154 cm
155 cm to 159 cm
160 cm to 164 cm
165 cm to 169 cm
170 cm to 179 cm

Then the class intervals are:

Group of height (h)	**Class interval in cm**
150 cm to 154 cm	$149.5 \leq h < 154.5$
155 cm to 159 cm	$154.5 \leq h < 159.5$
160 cm to 164 cm	$159.5 \leq h < 164.5$
165 cm to 169 cm	$164.5 \leq h < 169.5$
170 cm to 179 cm	$169.5 \leq h < 179.5$

The class intervals are not of equal width as the final class width is 10 cm.

The lower class boundary of the class interval is the smallest number that rounds up to the lower limit of the group, and the upper class boundary is the largest number which rounds down to the upper limit of the group.

The class limits correspond to the upper and lower values of the group that they relate to. In the table above, the first line has class limits of 150 cm and 154 cm but class boundaries of 149.5 cm and 154.5 cm.

Histograms

Histograms are used when you want to see the underlying shape of the distribution of the data – is it symmetrical, skewed, increasing, decreasing etc.? A histogram is a particular type of diagram which follows certain rules:

- The area (not the length) of the bar is proportional to the frequency.
- The bars are the same width as the class intervals (class width) and are bounded by the class intervals.
- Class intervals do not have to be equal so the bars can be of different widths.
- Bars are drawn on a continuous, linear, horizontal scale.
- There are no spaces between the bars unless a class interval contains no data.
- The vertical axis shows frequency density.

Frequency density is defined as

$$\frac{frequency}{class\ width}$$

If the data is spread symmetrically then the distribution is described as symmetrical.

Cumulative frequency

Cumulative frequency curves plot the cumulative frequency or running totals of the data. They are used when you want to be able to make simple estimates of certain statistics about the data set such as median, quartiles and range.

Cumulative frequency curves can be used for discrete or continuous data.

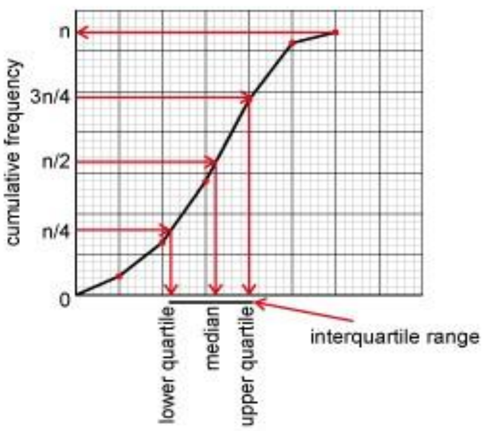
The running total is plotted against the upper class limit.

The curve is normally an elongated S shape.

Points can be joined with a curve or with straight lines.

The cumulative frequency is plotted on the vertical axis. If the highest value of the cumulative frequency is n then the median is the value read on the horizontal axis corresponding to $\frac{n}{2}$ on the vertical axis, and the quartiles are the 2

values on the horizontal axes corresponding to $\frac{3n}{4}$ and $\frac{n}{4}$ on the vertical axis. The interquartile range is the difference between the two quartile values.



If a cumulative frequency graph is drawn showing the number of questions, out of 20, answered correctly by a class of 36 pupils, the point (x, y) on the graph shows the number of pupils y who achieved x marks or fewer.

Discrete and continuous data

Are these data sets discrete or continuous?

Length of a leaf: is continuous as the length of the leaf can take any value reasonable for that type of leaf and not just particular values.

Number of insects caught in a trap: is discrete as the number is an integer.

Mass of kittens: is continuous as the mass can take any value reasonable for kittens and not just particular values.

Class intervals for continuous data

The heights of some tomato plants are measured correct to the nearest cm. Complete the table:

Class limits in cm	Class interval in m
50 to 59	$49.5 \leq h < 59.5$
60 to 69	
70 to 79	
80 to 89	
90 to 109	

As the measurements are correct to the nearest cm, then the lower boundary is the lowest number that corrects up to the lower limit. Therefore, on the first row the lower boundary is 49.5 because this is the lowest number that corrects to 50. Numbers below 59.5 will round down to 59, so 59.5 is the upper boundary. The upper boundary of one class is the lower boundary of the next class, and the inequality signs will be the same throughout for consistency. The lower boundary of the second class is 59.5, and the upper boundary is 69.5 as numbers below this round down to 69.

Class limits in cm	Class interval in cm
50 to 59	$49.5 \leq h < 59.5$
60 to 69	$59.5 \leq h < 69.5$
70 to 79	$69.5 \leq h < 79.5$
80 to 89	$79.5 \leq h < 89.5$
90 to 109	$89.5 \leq h < 109.5$

Histogram for continuous data

The table below, from the third example above, now has the frequencies added.

Complete the table by calculating the class widths and frequency densities.

Draw a histogram using the class intervals given to represent these data.

Class limits in cm	Class interval in cm	Frequency	Class width	Frequency density
50 to 59	$49.5 \le h < 59.5$	15		
60 to 69	$59.5 \le h < 69.5$	20		
70 to 79	$69.5 \le h < 79.5$	25		
80 to 89	$79.5 \le h < 89.5$	53		
90 to 109	$89.5 \le h < 109.5$	30		

To find the class widths, subtract the lower class boundary from the upper class boundary. The first one is 59.5 − 49.5 = 10. The rest are done similarly.

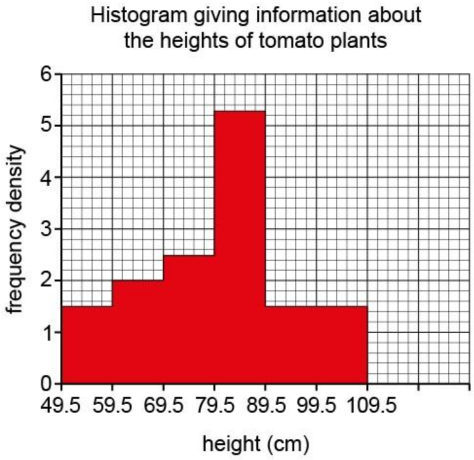
To find the frequency density, divide the frequency by the class width. The first one is 15 ÷ 10 = 1.5. The rest are done similarly.

Class limits in cm	Class interval in cm	Frequency	Class width	Frequency density
50 to 59	$49.5 \le h < 59.5$	15	10	1.5
60 to 69	$59.5 \le h < 69.5$	20	10	2
70 to 79	$69.5 \le h < 79.5$	25	10	2.5
80 to 89	$79.5 \le h < 89.5$	53	10	5.3
90 to 109	$89.5 \le h < 109.5$	30	20	1.5

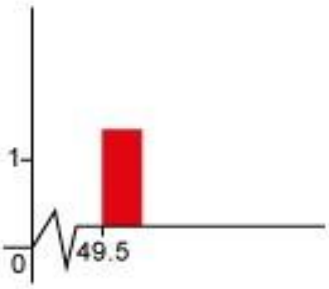
To draw the histogram, the axes must have clear labels and scales must be linear.

The vertical axis is labelled frequency density.

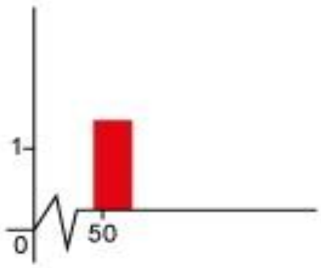
There is a large gap between 0 and 49.5, so the graph below has been drawn with the horizontal axis shown from 49.5 only.



You could also show the horizontal axis from zero with a zigzag line to show that the scale has been compressed:



You could also base the horizontal scale on multiples of 10 and start the bars at 0.5 below the 50 mark:



The bars are drawn with no spaces in between – unless one of the classes contains zero data – and a title is added.

Cumulative frequency

The frequency table shows the number of insects caught in a set of traps.

Fill in the cumulative frequencies and the associated intervals.

Draw the cumulative frequency graph.

Estimate from the graph the median number of insects caught in a trap.

Estimate the number of traps which contained more than 17 insects.

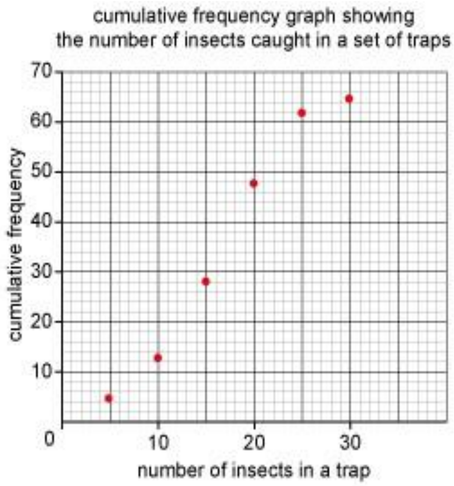
Number of insects (n)	**Class interval**	**Frequency**	**Cumulative frequency**	**Interval**
1 to 5	$0 < n \leq 5$	5		
6 to 10	$5 < n \leq 10$	8		
11 to 15	$10 < n \leq 15$	15		
16 to 20	$15 < n \leq 20$	20		
21 to 25	$20 < n \leq 25$	14		
26 to 30	$25 < n \leq 30$	3		

To work out the cumulative frequency, or running total, start by copying across the first frequency to the cumulative column.

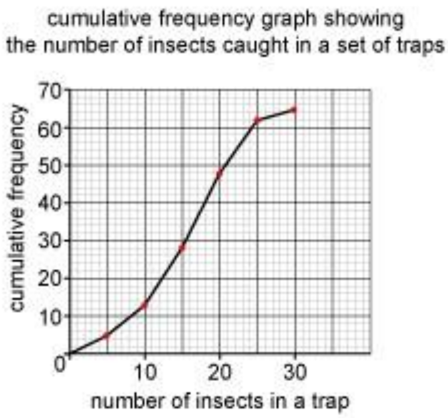
This is the number of traps that have caught up to 5 insects, so the interval is $0 < n \leq 5$ Then add this to the next frequency ($5 + 8 = 13$) and put the 13 in the cumulative column. This is the number of traps that have caught up to 10 insects so the interval is $0 < n \leq 10$ Continue in the same way to complete the table.

Number of insects (n)	**Class interval**	**Frequency**	**Cumulative frequency**	**Interval**
1 to 5	$0 < n \leq 5$	5	5	$0 < n \leq 5$
6 to 10	$5 < n \leq 10$	8	13	$0 < n \leq 10$
11 to 15	$10 < n \leq 15$	15	28	$0 < n \leq 15$
16 to 20	$15 < n \leq 20$	20	48	$0 < n \leq 20$
21 to 25	$20 < n \leq 25$	14	62	$0 < n \leq 25$
26 to 30	$25 < n \leq 30$	3	65	$0 < n \leq 30$

To plot the cumulative frequency graph, plot the cumulative frequency on the vertical axis against its upper class limit on the horizontal axis. The first 3 pairs of points are (5, 5), (10, 13) and (15, 28). The rest are done in the same way.

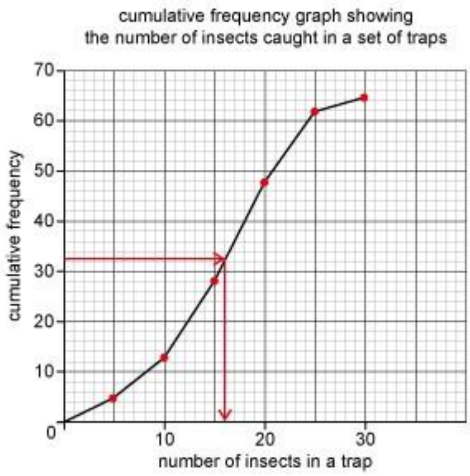


Now join the points with a freehand curve or straight lines. Include (0, 0) as this is the lower class limit:



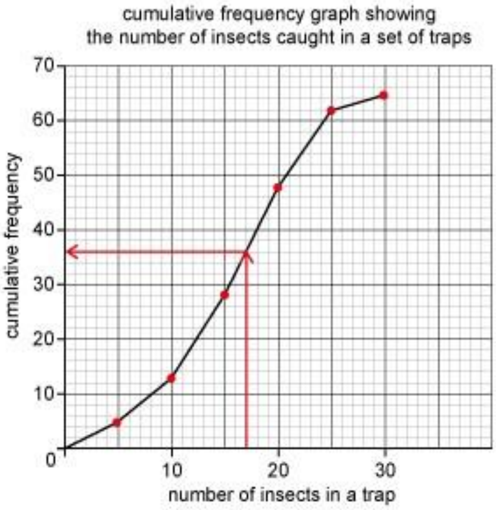
To find the median of the distribution, divide the total number of traps by 2, giving 32.5.

Draw a line across from 32.5 to the graph, and read the number of insects corresponding to this on the horizontal axis. In this case, approximately 16.



To find the number of traps with more than 17 insects, draw a line up from 17 to the curve and then read the number on the vertical axis: 36.

36 traps have 17 insects or fewer so 65 − 36 = 29 traps have more than 17 insects.


cumulative frequency graph showing
the number of insects caught in a set of traps
cumulative frequency
number of insects in a trap
0
10
20
30
40
50
60
70

M6.3

Calculate the *mean*, *mode*, *median* and *range* for ungrouped data.

Find the modal class; calculate estimates of the range, mean and median for grouped data, and understand why these are estimates.

Describe a population using statistics.

Make simple comparisons.

Compare data sets using like-for-like summary values.

Understand the advantages and disadvantages of summary values.

Calculate estimates of mean, median, mode, range, quartiles and interquartile range from graphical representation of grouped data.

Use the median and interquartile range to compare distributions.

Mean, mode, median and range for ungrouped data

The mean, median and mode are all averages.

The mean of a list of n numbers is the sum of the numbers divided by n. The mean does not have to be one of the numbers in the list.

$$mean = \frac{sum\ of\ all\ the\ values\ in\ the\ list}{number\ oof\ values\ in\ the\ list}$$

If the data is displayed in a frequency table, the sum of all of the values can be found either by adding all n numbers or by multiplying each number by its frequency and then adding the results.

If a list of n numbers is arranged in order of size, the median is the middle value of the list, which is the number in position $\frac{n+1}{2}$ in the list.

If n is odd, the median is a number in the list.

If n is even, the median falls between two numbers in the list, and is the average of those two numbers. This means that the median does not have to be one of the numbers in the list.

If the data is displayed in a frequency table then the median is found by counting through the table to find the number in position $\frac{n+1}{2}$

The mode of a list of numbers is the number that there is most of. There can be more than one mode or no mode (when all of the numbers have equal frequency). The mode has to be one of the numbers in the list.

If the data is displayed in a frequency table then the mode can be identified by finding the number(s) with the highest frequency.

The range is a measure of how spread out data is.

The range of a list of numbers is the positive difference between the largest number and the smallest number in the list.

Range = largest value − smallest value

If the data is displayed in a frequency table then the range can be found easily by identifying the largest value and smallest value and then subtracting. Care should be taken to calculate the range from the possible values, and not from the frequency column in the table.

Modal class for grouped data, estimates of range, mean and median for grouped data

The modal class for grouped data is the class which has the highest frequency.

Grouped data gives us intervals but not exactly which numbers are involved.

A row of a frequency table is shown for data which are integers. The only information given is that there are 9 numbers in the interval $6 < x \leq 10$ but there is no information as to which 9 numbers.

Class interval	**Frequency** (f)
$6 < x \leq 10$	9

The range, mean and median have to be estimated.

The range for data presented in a grouped frequency table is the difference between the lower limit of the lowest class interval and the higher limit of the highest class interval. It is an estimate because there is no information about whether or not these values are included in the data set.

To calculate an estimate of the mean, an estimate of the sum of all the data values is made. An estimate of the sum of the data values is calculated by using the mid-interval value for each class interval, multiplying the mid-interval values by the frequencies and then adding. It is an estimate because it assumes an even or symmetric distribution of data within the class intervals, which may not be the case.

Class interval	**Frequency** (f)	**Mid-interval value** (x)	f ×x
$6 < x \leq 10$	9	(6 + 10) ÷ 2 = 8	9 × 8 = 72

The median is the middle number when the data is listed in order of size. As the data in each class are not known exactly, an estimate must be made. If the median falls between the 3rd and 4th entry in the class interval shown then it is estimated as

$$6 + \left(\frac{3.5}{9} \times (10 - 6)\right) = 7\frac{5}{9}$$

For continuous data the same principles are applied:

Class interval	**Frequency** (f)	**Mid-interval value** (x)	f ×x
5.5 < x ≤ 10.5	9	(5.5 + 10.5) ÷ 2 = 8	9 × 8 = 72

If the median falls between the 3rd and 4th entry in the class interval shown then it is estimated as

$$5.5 + \left(\frac{3.5}{9} \times (10.5 - 5.5)\right) = 7\frac{1}{3}$$

Describing a population using statistics, making simple comparisons using statistics and comparing data sets using like-for-like summary values

The term 'statistic' refers to the various numbers that can be calculated about the data such as mean, median, mode, range, quartiles etc.

Considering the statistics of two or more data sets gives information about the similarities and differences of the data sets.

Summary values which describe central tendency are statistics such as mean, median and mode which define the centres which data tend to cluster around.

Summary values which describe the spread of the data are statistics such as range and interquartile range.

Advantages and disadvantages of summary data

The mean gives the arithmetic average and can be used for any numerical data; however, it is influenced by extreme values so may give a false view of the data.

The mode can be used for any type of data, not just numerical data, such as favourite colours. When the mode is used for numerical data, it is possible for the mode to be the lowest or highest data value and not a central value.

The median is not influenced by extreme values so shows a good indication of the central values, but it can only be used for data which can be ordered by size.

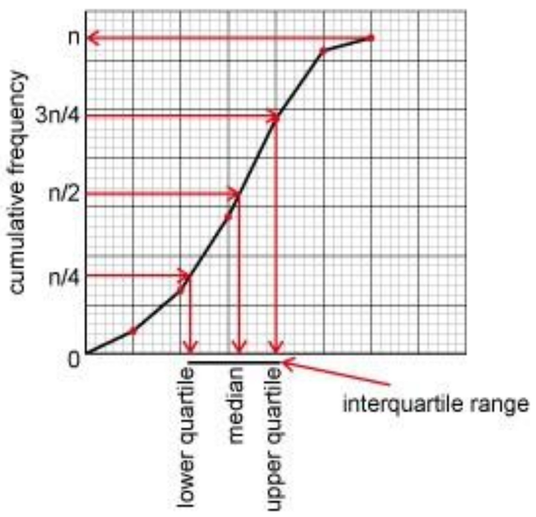
The range of the data gives a complete view of the spread of the data, but it is heavily influenced by extreme values.

The interquartile range shows the spread of the middle 50% of the data. It shows the positive difference between the lower quartile (the value $\frac{1}{4}$ of the way through the data when listed in order) and the upper quartile (the value $\frac{3}{4}$ of the way through the data when listed in order), so gives a good picture of the central data. The interquartile range is not affected by extreme values, but it does not give a complete picture of the range.

The advantages and disadvantages of summary data should be considered in terms of the context of the data.

Estimation of mean, median, mode, range, quartiles and interquartile range from graphical representation of grouped data

Estimating the median and quartiles from a cumulative frequency graph is covered in the section on graphical representations, but the diagram shows the summary.



The modal class or classes can be estimated, if the class widths are equal, as they are represented by the steepest line segment in the cumulative frequency graph.

Estimates of the mean and mode are better calculated from the original frequency table.

To calculate an estimate of the median from a histogram, first calculate the frequency represented by each block of the histogram. As frequency density is defined as $\frac{\text{frequency}}{\text{class width}}$ frequency is found by multiplying frequency density by class width. The frequencies are added to give the total number of data items (n). The estimated median is then item $\frac{n+1}{2}$ and its position and class are calculated in the same way as for grouped data. The quartiles are estimated in a similar way.

The mean is estimated by first multiplying the calculated frequency by the mid interval value for each of the bars, and adding the results. This result is divided by the total of the frequencies to get the estimated mean.

Using median and interquartile range to compare distributions

The median is a useful comparator of data sets as it gives the value of the middle item of the data, and is not as affected by exceptional or outlying values.

When the data is put in order of size, smallest first, the median and the quartiles split the ordered data into 4 regions each containing 25% of the data: data below the lower quartile – 25% data between the lower quartile and the median – 25% data between the median and the upper quartile – 25% data above the upper quartile – 25%

The interquartile range contains 50% of the ordered data.

If you are comparing the accuracy of two machines manufacturing set lengths of wire, you would want both the range and the interquartile range of the lengths to be as small as possible so the machines could be compared to see which had the smaller interquartile range.

Mean of ungrouped data

Find the mean of these numbers:

1193, 1194, 1196, 1198, 1199, 1201, 1203, 1204, 1205, 1207

Method 1

(1193 + 1194 + 1196 + 1198 + 1199 + 1201 + 1203 + 1204 + 1205 + 1207) ÷ 10 = 12 000 ÷ 10 = 1200

Method 2

For larger numbers which are close to each other, choose the smallest number (1193) and subtract it from every number in the list giving:

0, 1, 3, 5, 6, 8, 10, 11, 12, 14

The average is (0 + 1 + 3 + 5 + 6 + 8 + 10 + 11 + 12 + 14) ÷ 10 = 70 ÷ 10 = 7

Add this back to the 1193, giving 1193 + 7 = 1200 This gives an easier addition.

Method 3

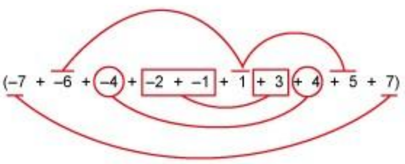
Another method of easing the arithmetic is to write the difference between these numbers and a central number. If you choose 1200 as the central number, the list becomes:

-7, -6, -4, -2, -1, 1, 3, 4, 5, 7

The average is (-7 + -6 + -4 + -2 + -1 + 1 + 3 + 4 + 5 + 7) ÷ 10 = 0

The average of the data is 1200 + 0 = 1200

Although this looks complicated, you do not have to add each number as the positive and negatives can be used to balance each other out:



The -7 and +7 make zero together, the -6 combines with +5 and +1 to make zero, -4 and +4 make zero and -2 and -1 and +3 make zero, giving a grand total of zero. This method does not always give such an easy answer, but it will simplify and speed up the calculation.

Median, mode, range

Find

the median

the mode

the range

of: 2, 5, 4, 1, 0, 6, 4, 8, 6, 9, 3, 8, 7, 1, 8, 6.

First put the numbers in order of size, smallest first:

0, 1, 1, 2, 3, 4, 4, 5, 6, 6, 6, 7, 8, 8, 8, 9.

(Count the number in both lists to make sure that you haven't missed one!)

There are 16 numbers in the list (zero is a number and must be counted), so the median is in position $\frac{16+1}{2}$ = 8.5. This means that you need to take an average of the 8th and 9th numbers in the list.

The median is $\frac{5+6}{2}$ = 5.5

There are three 6s and three 8s (and no numbers with higher frequencies), so there are 2 modes: 6 and 8.

The range is the difference between the largest and smallest number so is 9 − 0 = 9

Modal class for grouped data, estimates of range, mean and median for grouped data

The table below shows the age on their next birthday of the first 125 shoppers to arrive at a shopping centre one Tuesday morning.

Age	$10 \leq x \leq 19$	$20 \leq x \leq 29$	$30 \leq x \leq 39$	$40 \leq x \leq 49$	$50 \leq x \leq 59$	$60 \leq x \leq 69$	$70 \leq x \leq 79$
Frequency (f)	5	10	12	18	25	35	20

Write down the modal class.

Estimate the range of these data.

Estimate the mean of these data and give your answer correct to the nearest year.

Estimate the median of these data and give your answer correct to the nearest year.

The class which contains the most data is $60 \le x \le 69$ which has a frequency of 35.

The estimated range is found by subtracting the lowest age it is possible to have in these classes from the highest, giving 79 − 10 = 69.

To find the mean when you don't know the exact data, but only the class limits, you need to estimate the total of all the data. The mid-interval value is used as an approximation of the average value of the data in that interval.

Age	$20 \le x \le 29$	$10 \le x \le 19$	$30 \le x \le 39$	$40 \le x \le 49$	$50 \le x \le 59$	$60 \le x \le 69$	$70 \le x \le 79$
Frequency (f)	10	5	12	18	25	35	20
Mid-interval value (x)	24.5	14.5	34.5	44.5	54.5	64.5	74.5
f × x	245	72.5	414	801	1362.5	2257.5	1490

Adding all the f ×x values gives 6642.5

The total number of data values is the sum of the f values and is 125.

The mean is 6642.5 ÷ 125 = 53.14 which is 53 to the nearest year.

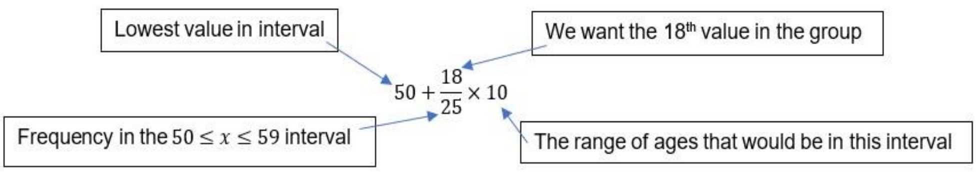
The median is the middle value of the data when the list is put in order of size. There are 125 ages in the data so the middle is the $\frac{1+125}{2}$ = 63rd age.

5 + 10 = 15

5 + 10 + 12 = 27

5 + 10 + 12 + 18 = 45 5 + 10 + 12 + 18 + 25 = 70 so the 63rd age lies in the interval $50 \le x \le 59$ and is the 63 − 45 = 18th age in that group.

The 18th age in the group is estimated as



This is 57.2 which is 57 to the nearest year.

Describing a population using statistics, making simple comparisons using statistics and comparing data sets using like-for-like summary values

Two factories, A and B, are making custom motorbikes. The number of motorbikes produced in each factory is recorded on a weekly basis. Each factory is closed for 4 weeks of each year and the output for those weeks is not recorded.

The statistics for production of motorbikes per week, over a working year, in the two factories are shown in the table below.

Statistic	Factory A motorbikes per week	Factory B motorbikes per week
Mean	14	15
Mode	14	13
Median	14	15
Lower quartile	13	13
Interquartile range	2	6
Range	4	11

What is the difference in the total number of motorbikes produced by each factory in a year?

Which of the factories has the biggest variability in the number of motorbikes produced in a week?

What are the upper quartile values for the output of the two factories?

A purchaser wishes to buy 11 motorbikes each week. Which factory consistently produced 11 or more motorbikes each week?

Which of the mean, median and mode has to be an actual data value?

The factories work for 48 weeks, so factory A makes 48 × 14 = 672 motorbikes and factory B makes 48 × 15 = 720motorbikes, so the difference is 720 − 672 = 48 motorbikes.

Factory B has a bigger range of data values so the most variation of number of motorbikes produced each week.c) Interquartile range = upper quartile value – lower quartile value

So for factory A, the upper quartile is 13 + 2 = 15 and for factory B it is 13 + 6 = 19

The lowest possible number (L) of motorbikes that factory A can produce is:

Highest number (H) of motorbikes produced minus the range which is H − 4

The upper quartile value is 15 so $H \geq 15$ so $H - 4 \geq 11$

Similarly, for factory B the minimum number is greater than or equal to $19 - 11 = 8$ So the buyer should choose factory A.

The mode is the only number which has to be an actual number of motorbikes produced. In this case, the median can fall between two values because there are an even number of data items, and the mean is the calculated arithmetic average so does not even have to be a whole number.

Advantages and disadvantages of summary data

The table shows the average yearly wage of workers in a factory. There are 6 cleaners paid the lowest wage, 36 machinists each paid twice the wage of a cleaner, 3 managers each paid 5 times the cleaners' wage and an owner paid 50 times the cleaners' wage.

	Cleaners	**Machinists**	**Managers**	**Owner**
Number	6	36	3	1
Pay	£x	£2x	£5x	£50x

Comment on the advantages and disadvantages of using summary data.

In this context the summary data consists of mean, median and mode as central measures, and range and interquartile range as measures of spread.

Looking at the table, it is obvious that the majority of workers earn £2x and that this is well represented by the median and mode.

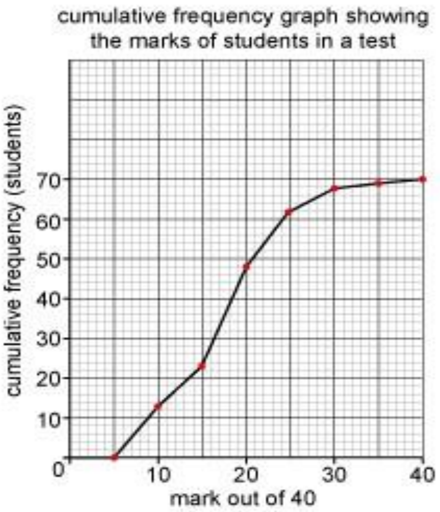
The mean has the disadvantage of having to include the exceptional values, so comes up with an average which is over 1.5 times the wage of the machinists, who are the majority of the employees. In this case, the mean is not a good estimator of the central trend.

The range is large and the median value tells us that there are some very high earners but has the disadvantage of not saying much about how the data is distributed. When considered with the mode and median, they give an idea of how far above the average wage the wage of the owner is.

In this case, the interquartile range has the disadvantage of giving no information about spread, but it does give a clear idea of the wage of the majority of workers.

Estimation of statistics from cumulative frequency graphs

The cumulative frequency graph shows the marks achieved by a class of 70 students in a History test which is marked out of 40.



Estimate the median.

Estimate the interquartile range.

Find the modal class.

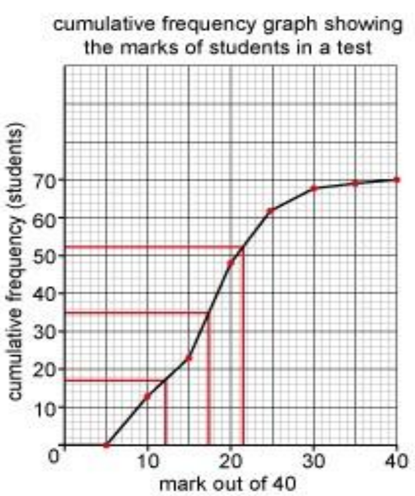
Find the range.

Explain how to calculate an estimate of the mean mark.

Draw in the median line at the 70 ÷ 2 = 35th student on the vertical axis. Read the value on the horizontal axis, which is approximately 17.5 marks.

The lower quartile is drawn from the mark of student number 70 × 0.25 = 17.5 and is approximately 12.25

The upper quartile is drawn from the mark of student number 70 × 0.75 = 52.5 and is approximately 21.5 The interquartile range is the difference between the upper and lower quartile values and is approximately 21.5 − 12.25 = 9.25



The modal class is the class with the steepest line if the intervals are equal. In this case it is the section of line representing marks between 15 and 20.

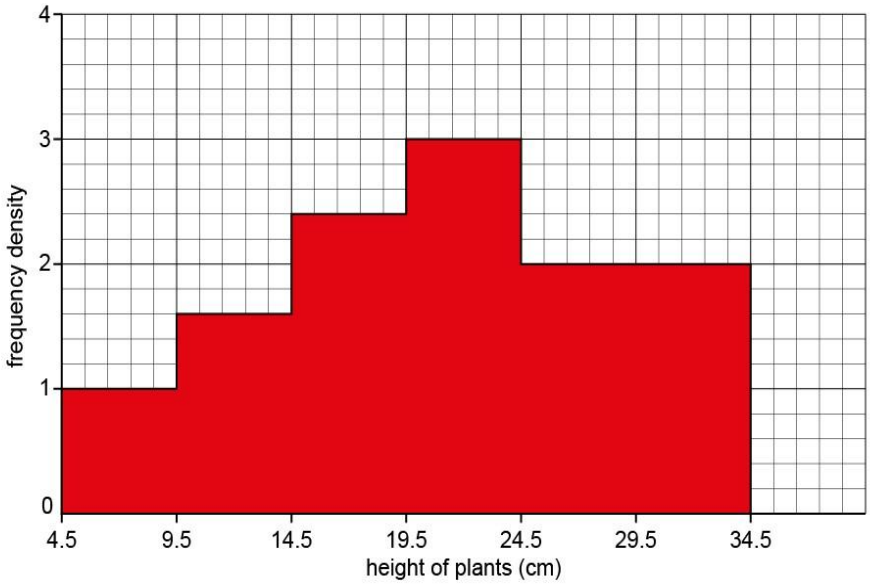
It is not possible to be confident of the highest and lowest marks because the cumulative frequency graph is drawn from grouped class intervals. The highest mark that might be in the data set is 40, and the lowest mark that might be in the data set is 5, so the range is 40 − 5 = 35

To estimate the mean, it is necessary to follow the same process as you would for finding the mean from a grouped frequency table. Find the mid interval value, multiply by the number of students in that interval, add the answers and then divide by the number of students. To do this from the graph, find the mid-interval values (7.5, 12.5 etc.) from the horizontal scale and then multiply each mid interval value by the difference of height across the interval (12-0, 23-12, etc.). This gives the number of students in that interval. Add the products and divide by the total number of students, which is 70.

Class interval	**Frequency**	**Mid-point**	**Frequency × midpoint**
$5 < m \le 10$	12	7.5	90
$10 < m \le 15$	11	12.5	137.5
$15 < m \le 20$	25	17.5	437.5
$20 < m \le 25$	14	22.5	315
$25 < m \le 30$	6	27.5	165
$30 < m \le 35$	1	32.5	32.5
$35 < m \le 40$	1	37.5	37.5
Total	**70**		**1215**

Estimation of statistics from histograms

The histogram shows the heights of some plants after 6 weeks of growth. The class boundaries are of the form 4.5 ≤ x < 9.5



What is the modal class?

What is the total number of plants?

Estimate the mean height of the plants. Give your answer correct to 1 decimal place.d) Estimate the range of the data.

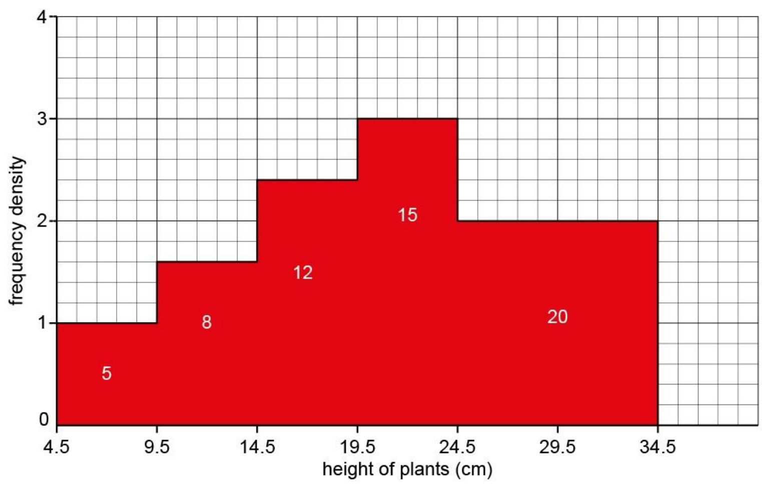
Which class interval contains the median?

In a histogram, the modal class can be defined as the class having the highest frequency density – the highest bar.

In this case it is 19.5 ≤ x < 24.5

To find the total number of plants, first find the frequency (the number of plants in each class).

As frequency density is $\frac{\text{frequency}}{\text{class width}}$ then frequency = frequency density × class width



Total number of plants = 5 + 8 + 12 + 15 + 20 = 60

To estimate the mean, multiply the frequencies by the mid interval values, as for frequency diagrams, and divide by the total frequency.

This gives: $(5 \times 7 + 8 \times 12 + 12 \times 17 + 15 \times 22 + 20 \times 29.5) \div 60 = 1255 \div 60 = 20.9$ correct to 1 d.p. d) The range is $34.5 - 4.5 = 30$

The median value of the 60 is the height of the middle plant when the plants are placed in order of size. Depending on how the class intervals are defined, this will be either the height of plant number 30 or 30.5 Adding the frequencies:

$5 + 8 = 13$

$5 + 8 + 12 = 25$

$5 + 8 + 12 + 15 = 40$

So, the 30th and 31st plants lie in the interval $19.5 \leq x < 24.5$

M6.4

Use and interpret scatter graphs of bivariate data.

Recognise correlation, and know that it does not indicate causation.

Draw estimated lines of best fit.

Interpolate and extrapolate apparent trends whilst knowing the dangers of so doing.

Definitions

Bivariate data is data involving two variables.

For example, the height of pupils in a class and their weight, the average growth of a set of tomato plants in a town in a day, and the average number of umbrellas sold in the same town that same day.

A scatter graph is the set of points on a graph drawn from a bivariate data set.

Scatter graphs are used to show a data set containing two variables and to see if there is any association (or relationship) between the two variables.

If one variable is believed to change in response to the other, the explanatory variable (the variable that is believed to cause variation in the other) goes on the horizontal axis and the other variable, the response variable (the one that changes in response to the change in the explanatory variable) goes on the vertical axis. In some cases, when we are looking for association, there is not a clear explanatory variable and response variable, and so the variables may be placed on the axes either way around.

If you are investigating the effect of height of pupils on their weight, the cause is height, so that goes on the horizontal axis. If you are comparing marks in two subjects then either subject can go on either axis.

Correlation

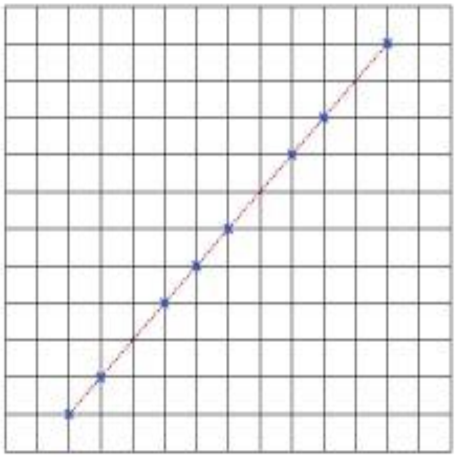
Correlation is a way of describing the relationship between the two variables. There are different types of correlation.

Linear correlation refers to straight line relationships between the two variables. It is also possible to have non-linear correlation where the two variables are associated but the data points form a curve which can take a wide variety of forms.

When considering correlation we can comment on:

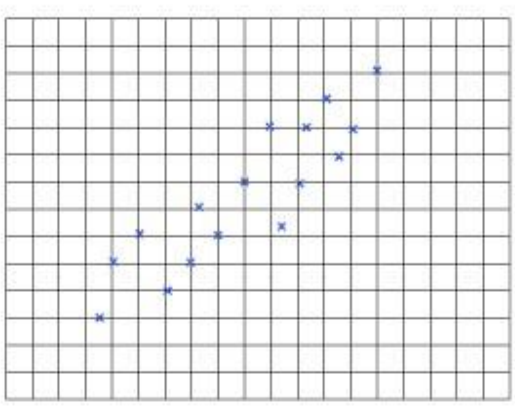
- whether the correlation is positive or negative;
- the strength of the correlation (perfect, strong, weak, no correlation).

Here is an example of perfect positive correlation. As one variable increases, the other increases (positive correlation) and the data points all lie on a straight line (perfect linear correlation).



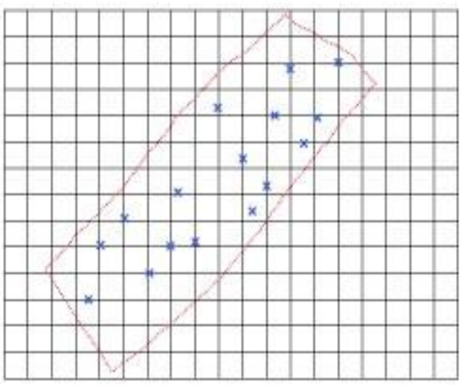
If the straight line has a negative gradient then the correlation is negative linear correlation.

Here is another example of positive correlation.

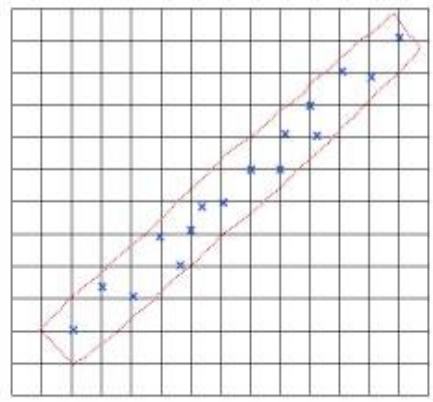


Note: If the points are not in a perfect line, it often helps to draw a rough rectangle around them to get a sense of shape. The narrower the rectangle, the stronger the correlation.

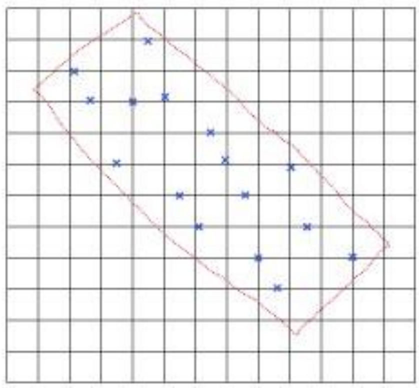
Weak positive correlation



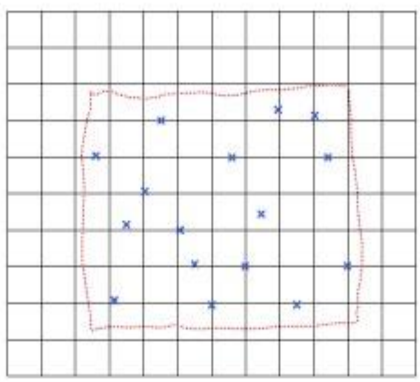
Strong positive correlation



Negative correlation



No correlation

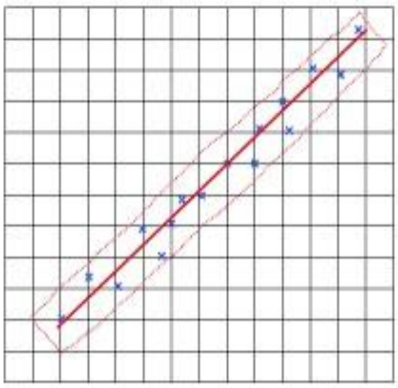


Correlation does not imply that a change in one variable is the cause of a change in the other variable. It may be that there is another variable that is causing the change in both of the variables measured, or it may be that we have an example of a spurious correlation – two variables that happen to show correlation by chance.

For example, the average growth of a set of tomato plants in a town in a day and the average number of umbrellas sold in the same town that same day show correlation, but the growth of the plants does not cause the public to buy more umbrellas. The cause is the level of rain!

Estimated lines of best fit

The line of best fit is a line drawn on the scatter graph which represents the general trend shown by the data. There should be approximately equal numbers of points above and below the line of best fit. The line of best fit may pass through some of the points but does not need to. It will be roughly parallel to the sides of the rectangle.

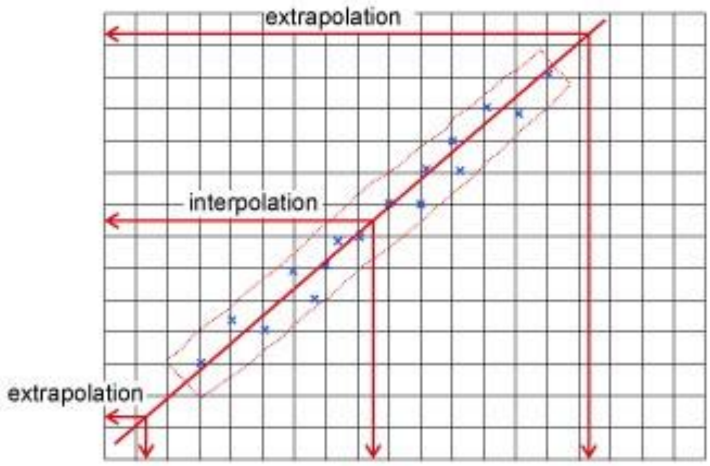


Interpolation and extrapolation

Interpolation and extrapolation are ways of estimating data not specifically in the given data set.

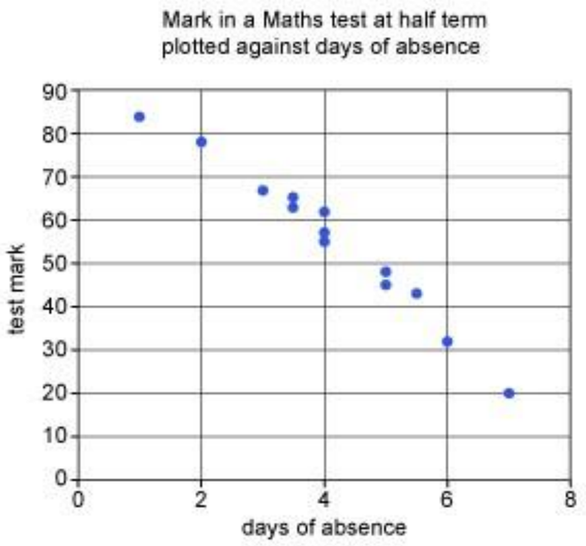
Interpolation is estimating points within the range of the given data set using the line of best fit. Extrapolation is estimating points outside the range of the given data set by extending the line of best fit.

Interpolation is generally more accurate than extrapolation as there is no real evidence of what happens outside the given data sets. The line of best fit may be part of a curve or the trend may change either inside or outside the data set.



Correlation

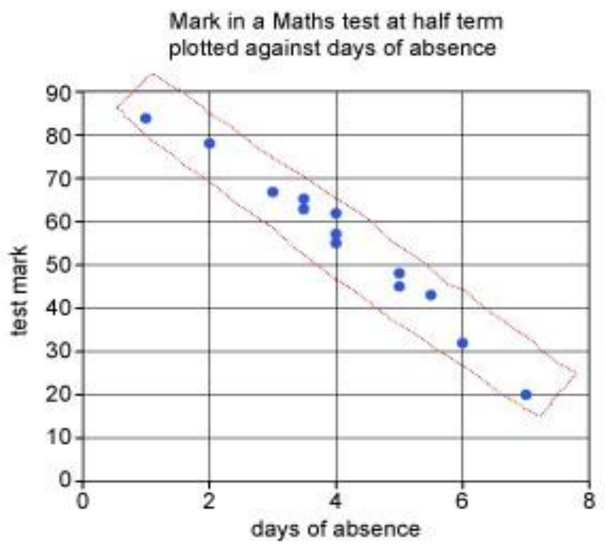
A Maths teacher feels that the number of days of absence a pupil has in a half term affects their marks in the half term test. She plots their marks against the number of days of absence.



Describe the correlation.

Do you think that the teacher's belief is justified?

Draw a rectangle around the points.



The rectangle is narrow and sloping downwards so there is strong negative correlation.

Always remember that correlation does not mean causation. Look for other factors which may affect the situation.

This strong correlation does indicate an association, but it cannot be stated that absence directly causes the lower marks as there may be other factors involved such as ability at Maths, having missed earlier work, illness or nerves on the day of the test.

Line of best fit, interpolation and extrapolation

The table shows the Physics and Chemistry marks (both out of 90) of 15 of the pupils in a class.

Chemistry	35	38	42	45	47	53	55	59	63	66	67	69	72	74	77
Physics	42	36	37	42	53	44	62	59	59	62	69	66	71	77	74

Draw the scatter graph for these data.

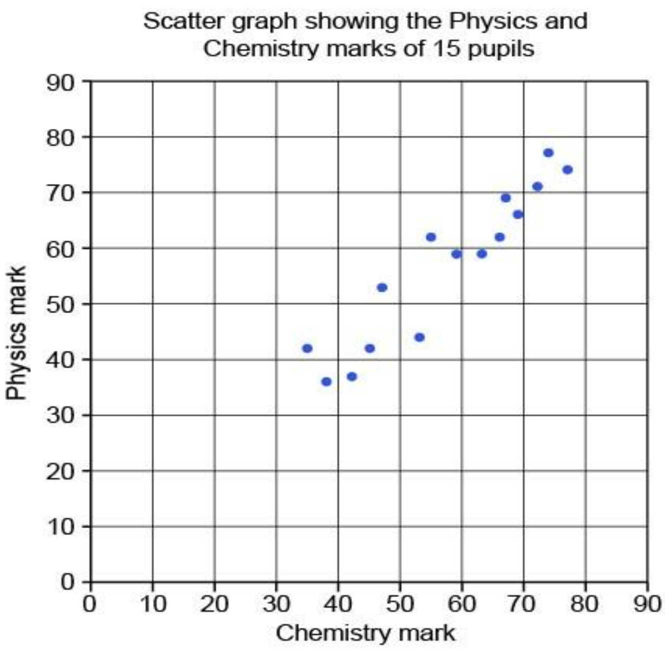
Describe the correlation.

Draw the estimated line of best fit.

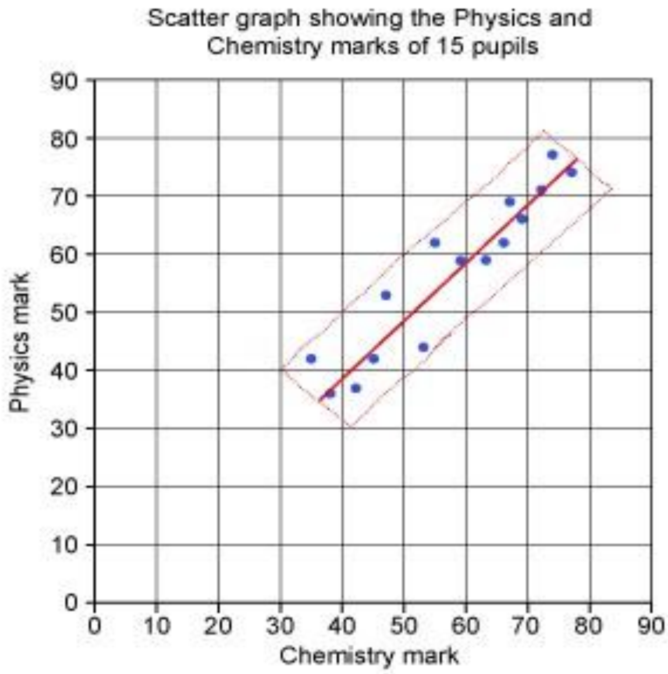
Another pupil in the class scores 50/90 in Chemistry. Estimate their Physics mark. Comment on the accuracy of this estimate. Did you use interpolation or extrapolation?

A pupil scores 85 in Physics. Estimate their Chemistry mark. Comment on the accuracy of this estimate. Did you use interpolation or extrapolation?

Plot the points in the usual way. As the Chemistry marks are given in order, it is easier to put Chemistry on the horizontal axis.



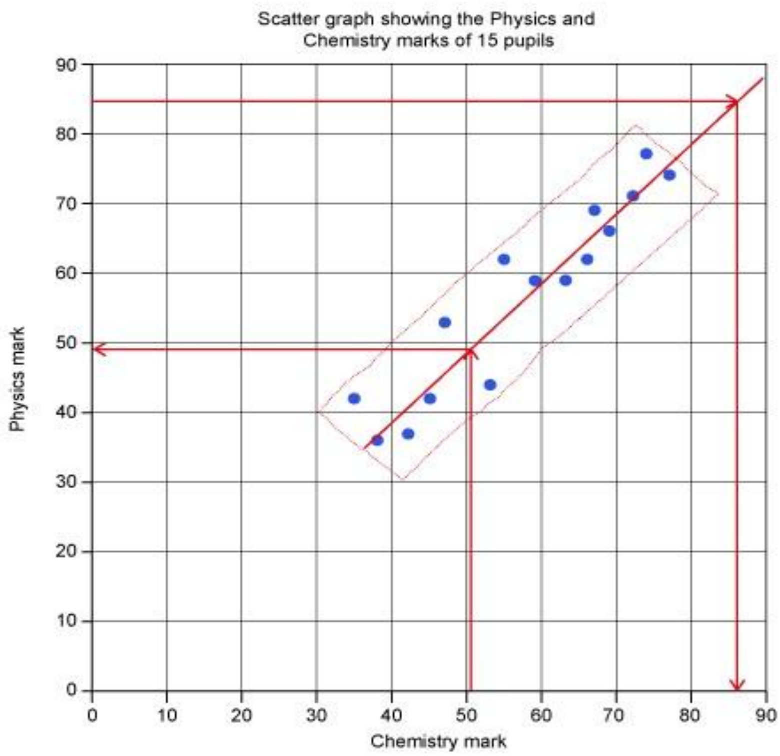
Drawing a rectangle around the points shows positive correlation, which demonstrates that there is a relationship between a pupil's marks in the two subjects.



The estimated line of best fit runs in approximately the same direction as the sides of the rectangle. Place it so that there is roughly the same number of points above and below the line.

To estimate the Physics mark corresponding to a Chemistry mark of 50, draw a line from the 50 on the Chemistry axis to the line of best fit and then draw across to read the Physics mark on the vertical axis. It is approximately 49. The method is shown on the diagram below.

As there is correlation, it is likely that the estimate is reasonable. It does not allow for the fact that this pupil may be much better, or much worse, at Physics than at Chemistry. This is interpolation as the point falls inside the data set.



To estimate the Chemistry mark corresponding to a Physics mark of 85, first extend the line of best fit so that a point with a vertical coordinate of 85 is available. The Chemistry mark is approximately 86. The method is shown on the diagram above.

Although there is correlation in the mark set given, there is no indication of what happens outside that set so the reliability of the line of best fit as an estimator outside the data set is not as certain as it is for points within the given data set. There is no guarantee that the data will follow the same pattern outside the given data set. It also does not allow for the fact that this particular pupil may be much better, or much worse, at Physics than at Chemistry.

This is extrapolation as the point falls outside the original data set.

M7. Probability

M7.1

Analyse the frequency of outcomes of probability experiments using tables and frequency trees.

The outcomes of probability experiments can be recorded and analysed using tables and frequency trees

Matthew has a spinner. The spinner has only a red section, a blue section and a yellow section.

Matthew thinks that the spinner may be biased.

He spins the spinner 50 times and records the colour that the spinner lands on.

The table below shows the outcomes of these 50 spins.

Colour	Red	Blue	Yellow	Total
Frequency	19	15	16	50

Use the outcomes of the 50 spins to estimate the probability of landing on each of the three colours.

Do you believe that the spinner is biased? Explain your answer.

Red $\frac{19}{20}$

Blue $\frac{15}{50}$

Yellow $\frac{16}{50}$

The relative frequencies for the different colours are all about the same: red 0.38, blue 0.3, yellow 0.32 when converted to decimals. This means that there is not sufficient evidence to suggest that the spinner is biased.

50 students are taking their driving test.

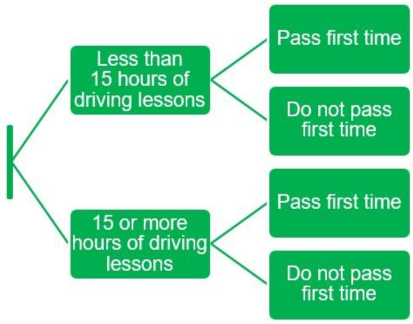
18 of the students have taken 15 or more hours of driving lessons.

11 of the students who have taken 15 or more hours of driving lessons pass first time.

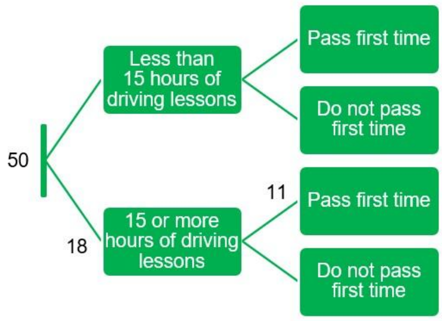
27 of the students do not pass first time.

Use a frequency tree to work out how many people have taken less than 15 hours of lessons and pass first time.

We start by drawing a frequency tree to represent the information that we are given in the question.



We then need to add the appropriate frequencies to the branches of the frequency tree.



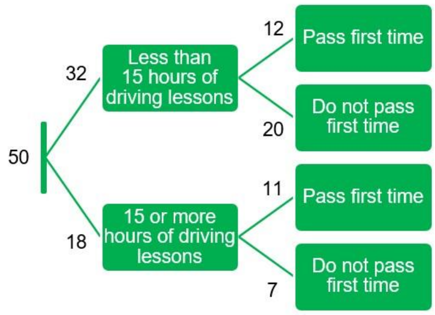
We can use the information that we have added so far, together with the additional information in the question, to complete the rest of the frequency tree.

As there are 50 students overall and 18 have had 15 or more hours of driving lessons, we can see that $50 - 18 = 32$ students must have had less than 15 hours of driving lessons.

Of the 18 students who have had 15 or more hours of driving lessons, we know that 11 passed first time. This means that $18 - 11 = 7$ students did not pass first time.

We are told that a total of 27 students did not pass first time. As 7 of these students had had 15 or more hours of driving lessons, we can see that $27 - 7 = 20$ of these students had had less than 15 hours of driving lessons.

Finally, a total of 32 students took less than 15 hours of driving lessons, of which 20 did not pass first time. This means that there must be $32 - 20 = 12$ students who have had less than 15 hours of driving lessons and passed first time.



There are therefore 12 students who had less than 15 hours of driving lessons and passed first time.

M7.2

Apply ideas of randomness, fairness and equally likely events to calculate expected outcomes of multiple future experiments.

Understand that if an experiment is repeated, the outcome may be different.

In probability, the sample space is the set of all possible outcomes of an experiment. For example, if the aim of the experiment was to toss a coin twice and record the result, then the sample space could be written as {HH, HT, TH, TT}.

An event is a subset of the sample space relating to an experiment. For example, when tossing a coin twice and recording the result, we could have an event 'one head is obtained'.

A fair die is one where there is an equally likely chance of landing on each of the faces. A fair spinner is one where there is an equally likely chance of spinning each of the possible outcomes.

We say that events are equally likely when there is an equal chance of getting each of the events. For example, if we toss a fair coin then the events 'heads' and 'tails' are equally likely.

We can use the theoretical probability of an event to calculate the expected outcomes of multiple future events. We do this by multiplying the theoretical probability by the number of trials of the experiment. For example, we can calculate the expected number of 6s when a fair six-sided die is rolled 30 times ($\frac{1}{6} \times 30 = 5$).

When the theoretical probability is used to calculate the expected outcomes of multiple future events, it does not mean that if the probability experiment is carried out the expected outcome will be seen. For example, the expected number of 6s when a fair six-sided die is rolled 30 times is 5; however, it is possible to roll fewer or more 6s than this.

If a fair six-sided die is rolled 90 times, how many times would it be expected to land on: a) 5?

an even number?

a square number?

The possible outcomes of rolling a fair six-sided die are 1, 2, 3, 4, 5, 6. Each of these outcomes is equally likely and so has a probability of $\frac{1}{6}$

P(land on 5) = $\frac{1}{6}$

Expected number of 5s in 90 rolls = $\frac{1}{6}$ × 90 = 15

P(land on an even number) = P(land on 2) + P(land on 4) + P(land on 6)

= $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

Expected number of even numbers in 90 rolls = $\frac{1}{2}$ × 90 = 45

P(land on a square number) = P(land on 1) + P(land on 4)

= $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

Expected number of square numbers in 90 rolls = $\frac{1}{3}$ × 90 = 30

A fair four-sided spinner has one section coloured red, one section coloured yellow, one section coloured green and one section coloured blue.

If the spinner is spun 100 times, how many times would 'yellow' be expected?

Explain why the actual number of 'yellow' in 100 spins might not be exactly the number expected.

Expected number of 'yellow' in 100 spins = P(land on yellow) × 100

= $\frac{1}{4}$ × 100

= 25

The actual number of spins landing on 'yellow' will not necessarily match the expected number based on the theoretical probability. Any number of 'yellow' from 0 to 100 might occur, although total numbers of 'yellow' closer to 25 are more likely.

M7.3

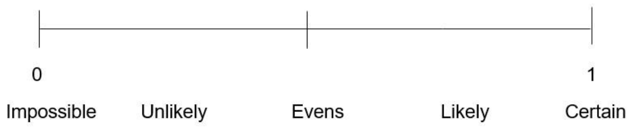
Relate relative expected frequencies to theoretical probability, using appropriate language and the ‘0 to 1’ probability scale.

The relative expected frequency can be used to calculate the theoretical probability of an event.

$$probability = \frac{number\ of\ favourable\ outcomes}{number\ of\ possible\ outcomes}$$

Probabilities may be given as fractions, decimals or percentages.

Probabilities can be shown on a probability scale from 0 to 1 (or 0% to 100%).



Rolling a fair six-sided die

A fair six-sided die is rolled.

Work out the probability that the die lands on:

number 3

an even number.

a prime number.

a square number.

$$P(\text{land on 3}) = \frac{\text{number of ways the die can land on 3}}{\text{number of possible outcomes}} = \frac{1}{6}$$

$$P(\text{land on an even number}) = \frac{\text{number of ways the die can land on an even number}}{\text{number of possible outcomes}} = \frac{3}{6} = \frac{1}{2}$$

$$P(\text{land on a prime}) = \frac{\text{number of ways the die can land on a prime number}}{\text{number of possible outcomes}} = \frac{3}{6} = \frac{1}{2}$$

This is because the prime numbers between 1 and 6 are 2, 3, 5.

$$P(\text{land on a square}) = \frac{\text{number of ways the die can land on a square number}}{\text{number of possible outcomes}} = \frac{2}{6} = \frac{1}{3}$$

This is because the square numbers between 1 and 6 are 1 and 4.

M7.4

Apply the property that the probabilities of an exhaustive set of outcomes sum to one.

Apply the property that the probabilities of an exhaustive set of mutually exclusive events sum to one.

A set of events is said to be exhaustive if at least one of the events must occur.

Events are said to be mutually exclusive if no more than one of the events can occur at any time.

It is possible for a set of events to be both exhaustive and mutually exclusive. For example, if a fair six-sided die is rolled, the two events 'land on an even number' and 'land on an odd number' are both exhaustive (as at least one of the events must occur) and mutually exclusive (as the die cannot land on an odd number and an even number simultaneously).

The probabilities of an exhaustive set of mutually exclusive events sum to one. We can use this to find unknown probabilities.

Two events A and not A will be exhaustive and mutually exclusive.

This means that P(A) + P(not A) = 1

We can use this to find the probability of an event not occurring if we know the probability of the event occurring.

The probability of an event not occurring

If a card is picked at random from a deck, the probability of it being an ace is $\frac{1}{13}$

What is the probability that it is not an ace?

The probability of rain tomorrow is 0.3.What is the probability of no rain tomorrow?

Marta is going to pick a book at random from her local library. The probability that the book is a fiction book is 65%. What is the probability that the book is non-fiction?

P(A) + P(not A) = 1

P(ace) = $\frac{1}{13}$

P(ace) + P(not ace) = 1

P(not ace) = 1 − $\frac{1}{13}$ = $\frac{12}{13}$

P(rain) = 0.3

P(rain) + P(not rain) = 1 P(not rain) = 1 − 0.3 = 0.7

P(fiction) = 65%

P(fiction) + P(not fiction) = 100%

P(not fiction) = 100% − 65% = 35%

Finding an unknown probability

A bag contains red counters, green counters and yellow counters only.

A counter is picked at random from the bag.

The probability of picking a red counter is 0.35; the probability of picking a yellow counter is 0.2.

Work out the probability of picking a green counter.

As the counters are red or yellow or green, then the events picking a red counter, picking a green counter and picking a yellow counter are mutually exclusive as none of the counters is more than one colour.

As there are only counters of these three colours in the bag, the events are exhaustive as any counter picked out must be one of these three.

P(red counter) + P(green counter) + P(yellow counter) = 1

0.35 + 0.2 + P(yellow counter) = 1

P(yellow counter) = 1 − 0.35 − 0.2 = 0.45

Finding an unknown probability

Marisa has a biased spinner.

When the spinner is spun, it can land on red or on green or on blue or on yellow or on purple.

The probability that the spinner lands on red is $\frac{1}{5}$

The probability that the spinner lands on blue is $\frac{1}{8}$

The probability that the spinner lands on yellow is $\frac{3}{10}$

The ratio of the probability of landing on green to the probability of landing on purple is 3 : 2

Work out the probability of the spinner landing on green when it is spun.

P(red) + P(green) + P(blue) + P(yellow) + P(purple) = 1

$\frac{1}{5}$+ P(green) + $\frac{1}{8}$ + $\frac{3}{10}$ + P(purple) = 1

$\frac{8}{40}$ + P(green) + $\frac{5}{40}$ + $\frac{12}{40}$ + P(purple) = 1

$\frac{25}{40}$ + P(green) + P(purple) = 1

P(green) + P(purple) = 1 − $\frac{25}{40}$ = $\frac{15}{40}$

We know that the probability of green and the probability of purple are in the ratio 3 **:** 2, so we divide $\frac{15}{40}$ in this ratio:

$\frac{15}{40}$ ÷ (3 + 2) = $\frac{3}{40}$

P(green) = 3 × $\frac{3}{40}$ = $\frac{9}{40}$

M7.5

Enumerate sets and combinations of sets systematically, using tables, grids, Venn diagrams and tree diagrams. Candidates are not expected to know formal set theory notation.

In probability questions it is often important to be able to list the possible outcomes of an experiment systematically and to organise information using tables, grids, Venn diagrams and tree diagrams to answer questions.

Listing sets

For example, the set of vowels is {a, e, i, o, u} and the set containing the first 10 prime numbers is {2, 3, 5, 7, 11, 13, 17, 19, 23, 29}.

Listing combinations of sets

Let A be the set containing the first 10 prime numbers.

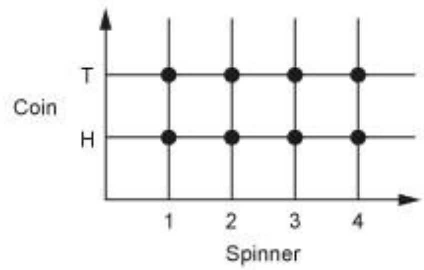
Let B be the set of odd numbers from 1 to 20.

The members of both set A and set B are the numbers that are both in the first 10 prime numbers, and are an odd number between 1 and 20. This is {3, 5, 7, 11, 13, 17, 19}.

Using grids

A grid can be drawn with dots representing possible outcomes. In order to answer a probability question, the relevant points on the grid are identified.

For example, an unbiased coin is flipped and a four-sided spinner with sides numbered 1, 2, 3, 4 is spun. The diagram below shows the possible outcomes.



Using tables

If two unbiased dice are rolled and the pairs of scores recorded, the possible outcomes can be shown in a sample space (sometimes called a possibility space) diagram:

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Using Venn diagrams

A Venn diagram uses circles (or other shapes) to show members of sets.

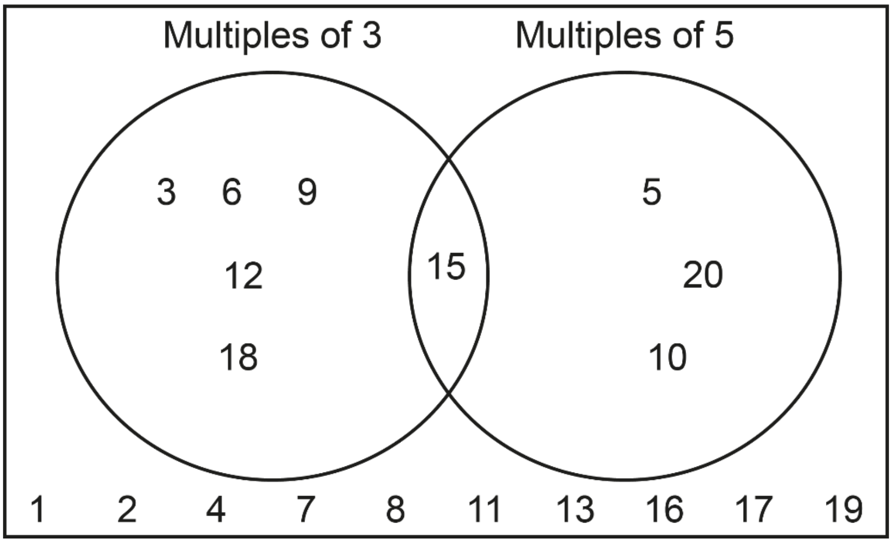
A member of a set is written inside the circle for that set.

A member of two (or more) sets is shown in the overlap of the two circles.

The area surrounding the circles shows the elements that are not members of either (all) of the sets.

For example, this Venn diagram is for integers from 1 to 20.

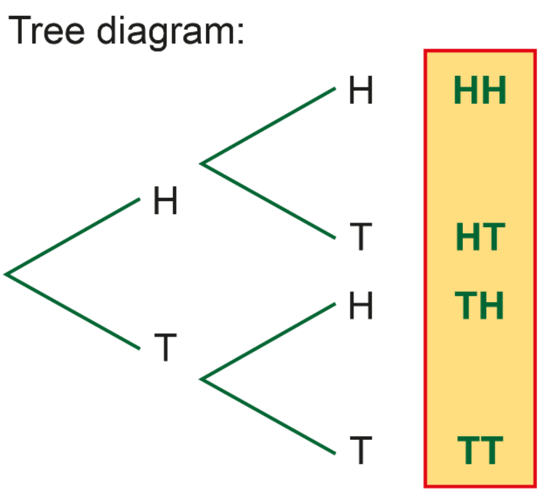
The sets are the multiples of 3 and the multiples of 5.



Using tree diagrams

A tree diagram can be used to list a set of outcomes.

For example, listing the set of outcomes of two coin flips:



The set of outcomes of two coin flips is {HH, HT, TH, TT}.

Listing sets

List the set of the first 10 multiples of 6.

List the set of the first 10 multiples of 8.

{6, 12, 18, 24, 30, 36, 42, 48, 54, 60}

{8, 16, 24, 32, 40, 48, 56, 64, 72, 80}

Listing combinations of sets

Let A be the set of the first 10 multiples of 6.

Let B be the set of the first 10 multiples of 8.

List the numbers that are members of both A and B.

Set A is {6, 12, 18, 24, 30, 36, 42, 48, 54, 60}.

Set B is {8, 16, 24, 32, 40, 48, 56, 64, 72, 80}.

We need to identify numbers that are members of both set A and set B. There are two numbers that are in both of the lists: 24 and 48.

Using grids

An unbiased coin is flipped and a fair five-sided spinner with sides numbered 1, 2, 3, 4, 5 is spun.

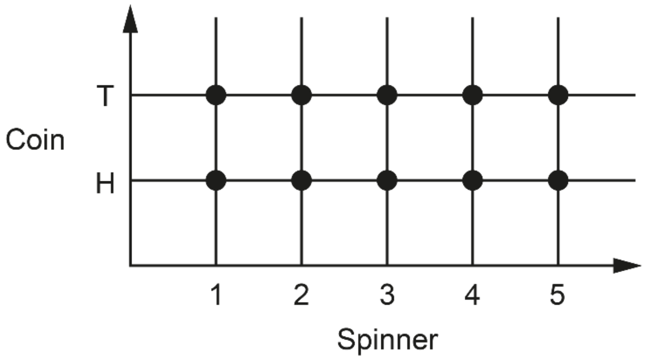
Draw a grid to represent the possible outcomes.

Work out the probability of either a head or an odd number or both.

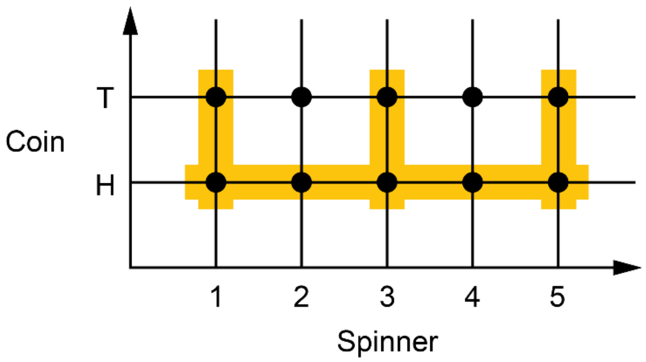
Work out the probability of a tail but not an odd number.

We start by drawing a set of axes with the possible outcomes of flipping a coin on one and the possible outcomes of spinning the spinner on the other.

We can then draw our grid of possible outcomes for the coin being flipped and the spinner being spun.



To find the probability of either a head or an odd number or both, we need to identify the number of outcomes with a head or an odd number or both. We can do this from the grid.

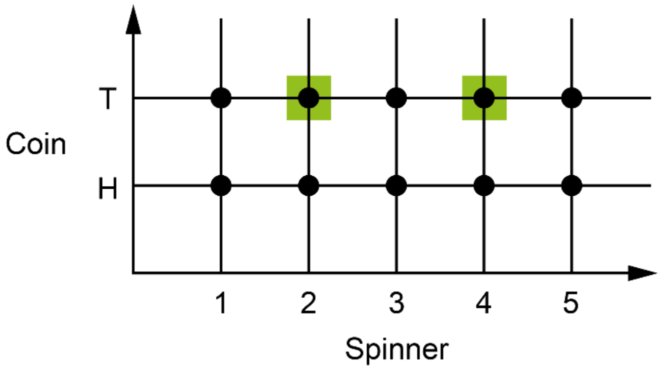


There are 8 outcomes that have a head or an odd number or both.

There are 10 possible outcomes in total.

P(head or odd number or both) = $\frac{8}{10} = \frac{4}{5}$

To find the probability of a tail but not an odd number, we need to identify the number of outcomes with a tail but not an odd number. We can do this from the grid.



There are 2 outcomes that have a tail but not an odd number.

There are 10 possible outcomes in total.

P(tail but not an odd number) = $\frac{2}{10} = \frac{1}{5}$

Using tables

Two four-sided unbiased dice have sides numbered 1, 2, 3 and 4.

Draw a sample space diagram to show the set of all possible outcomes of the two dice rolls.

How many of the possible outcomes have the same score on both dice?

How many of the possible outcomes have a total score of 5 or more?

How many of the possible outcomes have a total score of 1?

	1	**2**	**3**	**4**
1	(1,1)	(1,2)	(1,3)	(1,4)
2	(2,1)	(2,2)	(2,3)	(2,4)
3	(3,1)	(3,2)	(3,3)	(3,4)
4	(4,1)	(4,2)	(4,3)	(4,4)

The outcomes with the same score on both dice are (1, 1), (2, 2), (3, 3), (4, 4). There are a total of 4 possibilities.

The outcomes that have a total score of 5 or more are highlighted in the copy of the table below.

	1	**2**	**3**	**4**
1	(1,1)	(1,2)	(1,3)	(1,4)
2	(2,1)	(2,2)	(2,3)	(2,4)
3	(3,1)	(3,2)	(3,3)	(3,4)
4	(4,1)	(4,2)	(4,3)	(4,4)

There are 10 pairings.

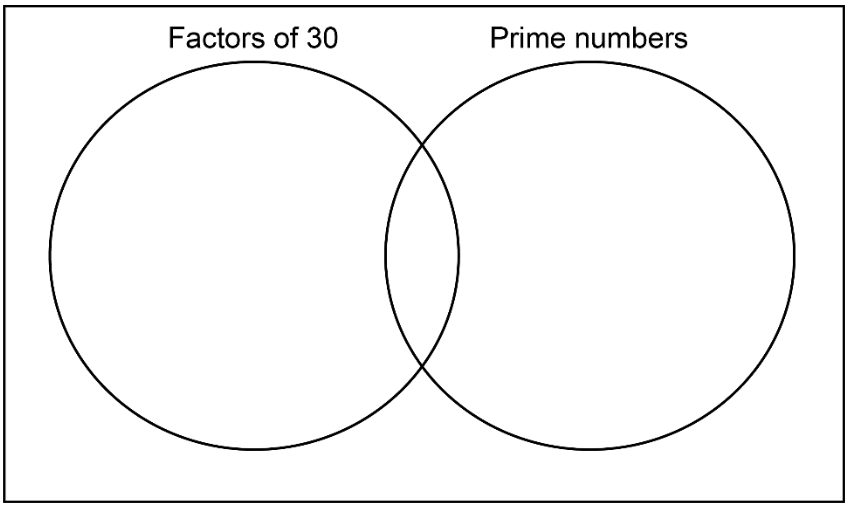
Note: You could also have drawn a sample space that showed the total scores to answer this part of the question. An example is shown below.

	1	**2**	**3**	**4**
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

None of the pairings have a total score of 1. In fact, the lowest possible score is 2.

Using Venn diagrams

Complete the Venn diagram below for the integers from 1 to 30.



How many of the integers from 1 to 30 are factors of 30?

How many of the integers from 1 to 30 are prime numbers and factors of 30?

What is the probability of an integer chosen at random from 1 to 30 being both prime and a factor of 30?

What is the probability of an integer chosen at random from 1 to 30 being prime but not a factor of 30?

What is the probability that an integer chosen at random from 1 to 30 is neither prime nor a factor of 30?

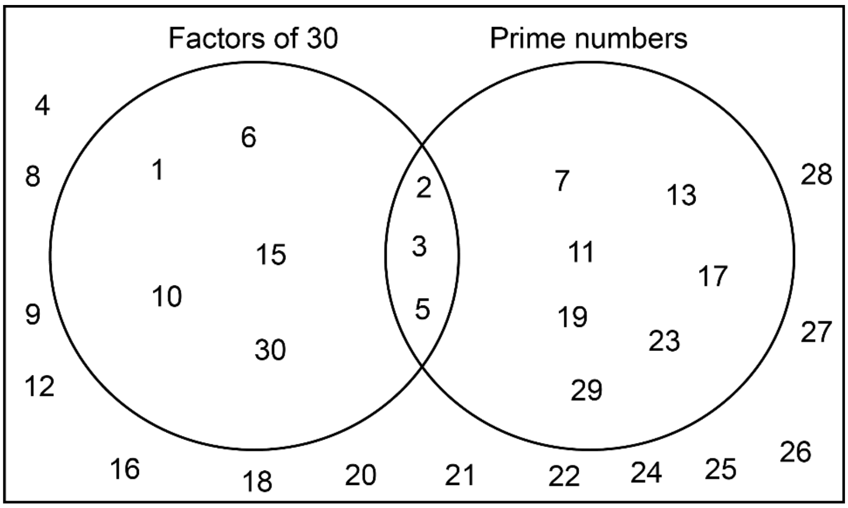
Consider each of the integers from 1 to 30 in turn.

If the number is a factor of 30, it is placed in the left-hand circle.

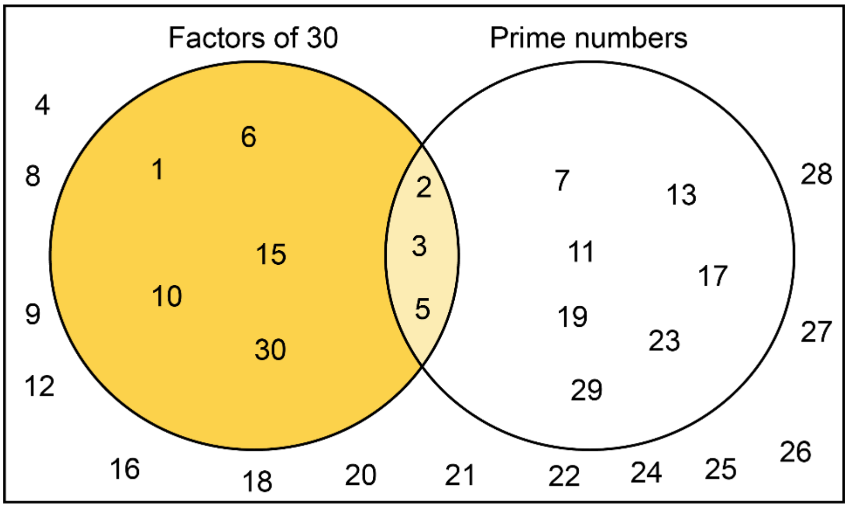
If the number is a prime number, it is placed in the right-hand circle.

If the number is both a factor of 30 and a prime number, it should be placed in the overlap.

If the number is neither a factor of 30 nor a prime number, it should be placed in the region outside the two circles.

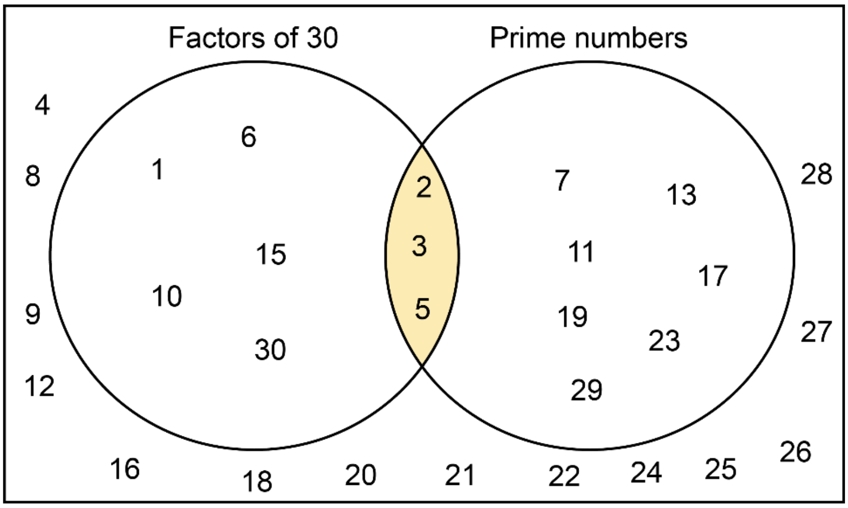


The integers between 1 and 30 that are factors of 30 are contained within the left-hand circle. This includes the overlap region.



There are 8 integers between 1 and 30 that are factors of 30.

The integers between 1 and 30 that are factors of 30 and prime numbers are contained within the overlap region.

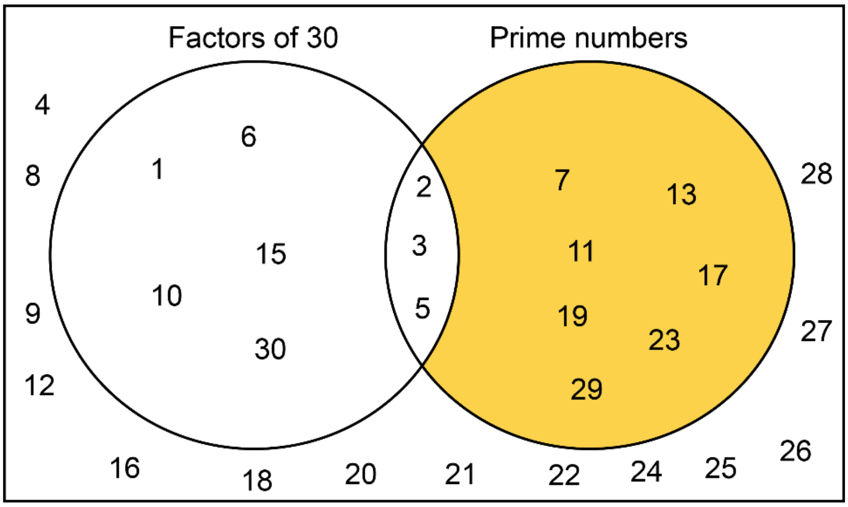


There are 3 integers between 1 and 30 that are factors of 30 and prime numbers.

The probability of an integer chosen at random from 1 to 30 being both prime and a factor of 30 =

$$\frac{\text{number of integers between 1 and 30 that are both prime and a factor of 30}}{\text{total number of integers between 1 and 30}} = \frac{3}{30} = \frac{1}{10}$$

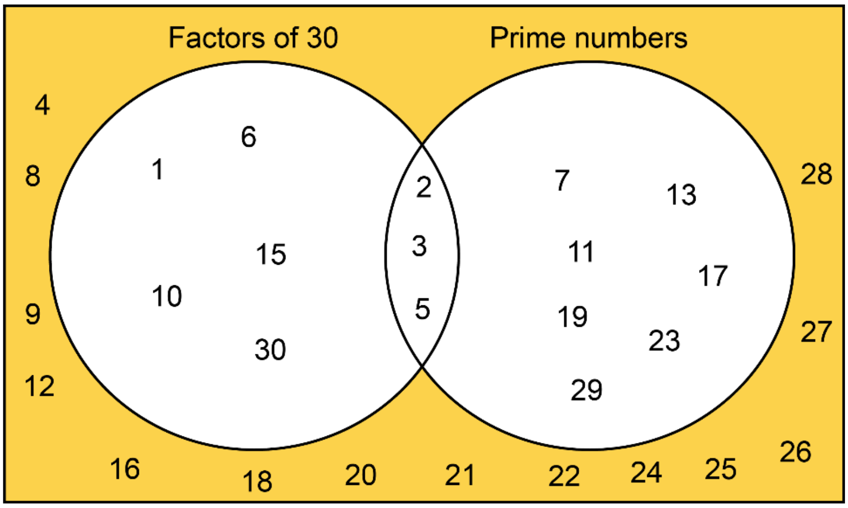
To find the probability of one of these integers being chosen at random being a prime number but not a factor of30, we start by identifying how many integers between 1 and 30 are prime but not a factor of 30. These are the integers that are included in the right-hand circle, but are not in the overlap as shown below.



There are 7 numbers that are prime but not a factor of 30.

P(integer between 1 and 30 being prime, but not a factor of 30) = $\frac{7}{30}$

To find the probability of one of these integers being chosen at random being neither prime nor a factor of 30, we start by identifying how many integers between 1 and 30 are neither prime nor a factor of 30. These are the integers that are not contained within either the left-hand or right-hand circles as shown below.

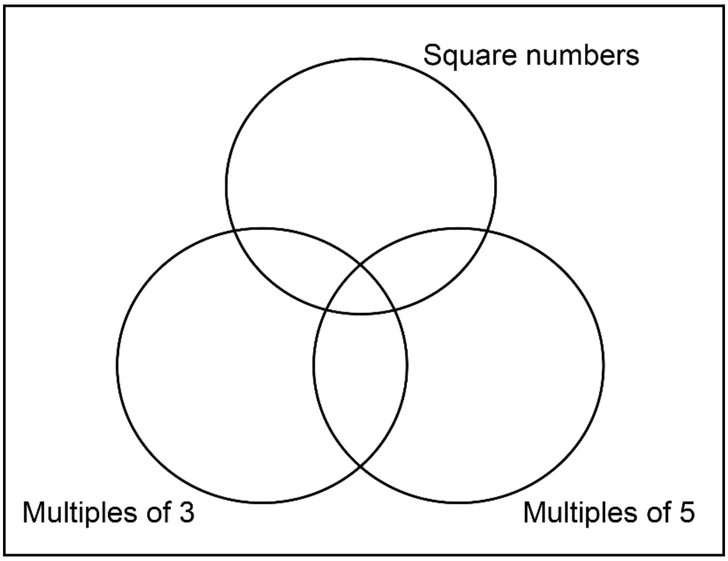


There are 14 integers that are neither prime nor a factor of 30.

P(integer between 1 and 30 that is neither prime nor a factor of 30) = $\frac{14}{30} = \frac{7}{15}$

Using Venn diagrams

Complete the Venn diagram below for odd numbers from 0 to 50.



How many of the odd numbers from 0 to 50 are a multiple of 3 and a multiple of 5?

How many of the odd numbers from 0 to 50 are a multiple of 3 or a multiple of 5?

One of the odd numbers from 0 to 50 is to be chosen at random.

What is the probability that the number chosen is:

a square number, a multiple of 3 and a multiple of 5?

a square number?

a square number, but not a multiple of 3 or 5?

a square number, but not a multiple of 3?

not a square number, a multiple of 3 or a multiple of 5?

If the range of numbers were increased, what would be the first odd number that would be in the central region of the Venn diagram (where all three circles overlap)?

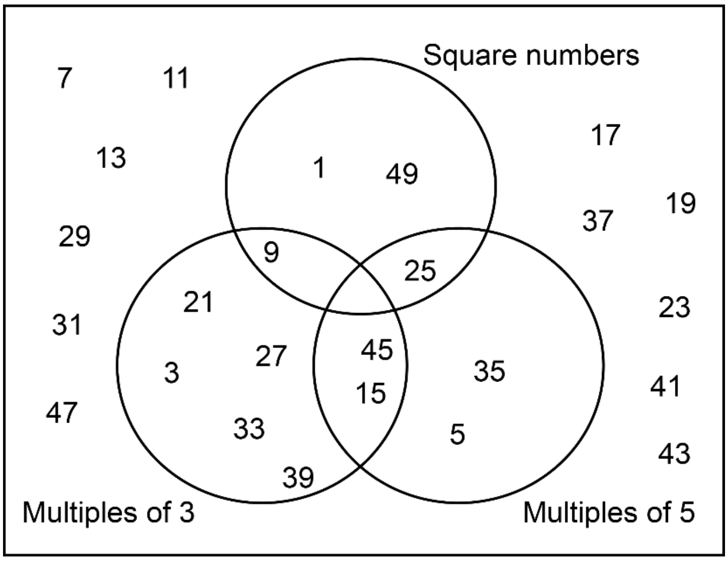
Consider each of the odd numbers from 0 to 50 in turn. If the number is a multiple of 3, it is placed in the bottom left circle.

If the number is a multiple of 5, it is placed in the bottom right circle.

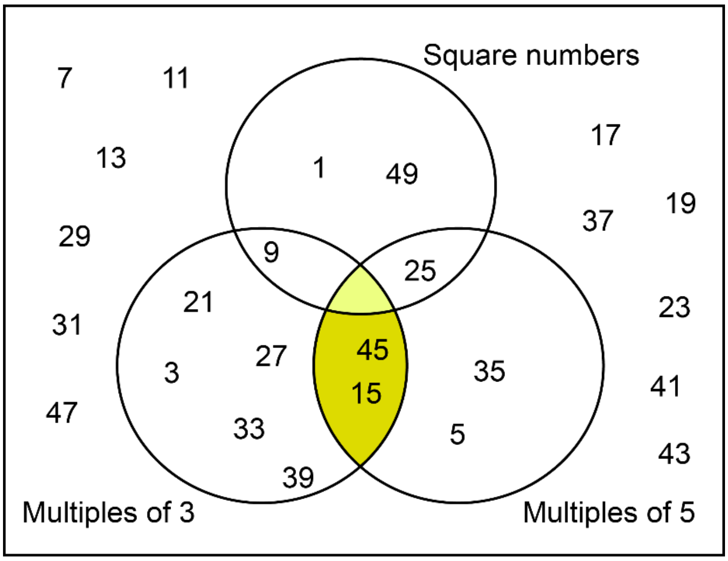
If the number is a square number, it is placed in the top circle.

If the number is in more than one category, it is placed in the region of overlap between the two (or three) circles.

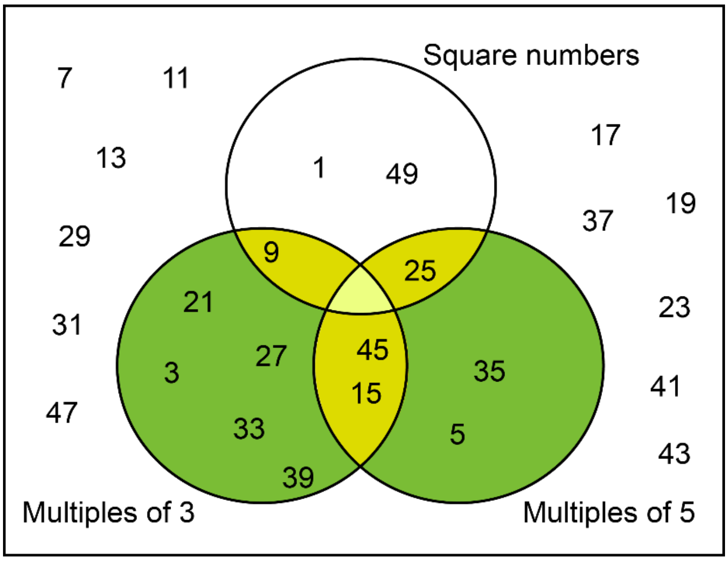
If the number is not a multiple of 3, a multiple of 5 or a square number, then it is placed in the region outside the circles.



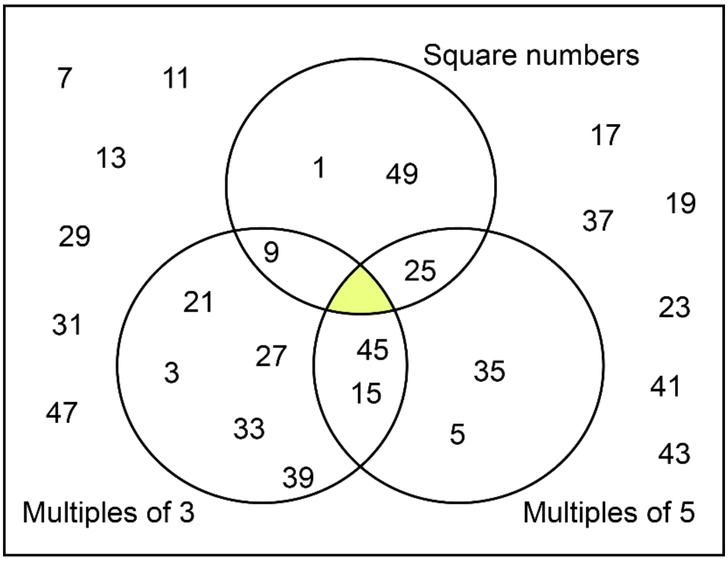
The odd numbers between 0 and 50 that are a multiple of 3 and a multiple of 5 are the ones in the overlap between the two circles. There are 2 numbers in this region of the Venn diagram.



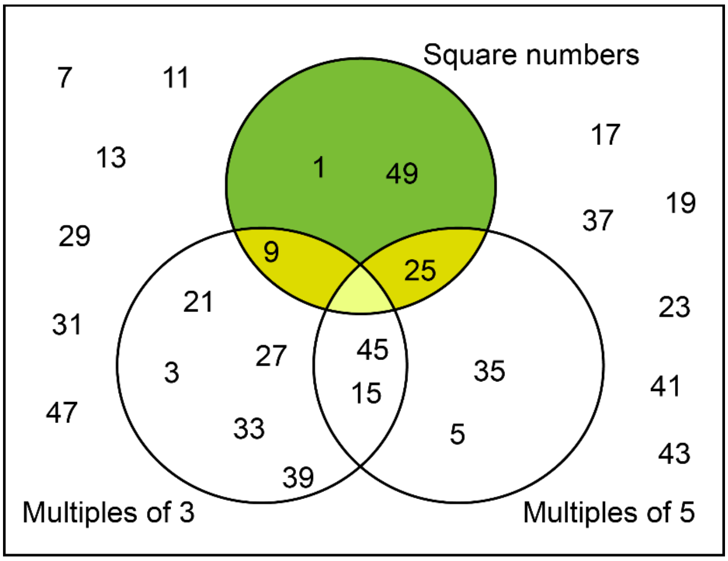
The odd numbers between 0 and 50 that are a multiple of 3 or a multiple of 5 are any values contained in either of these two circles. There are 11 numbers in this region of the Venn diagram.



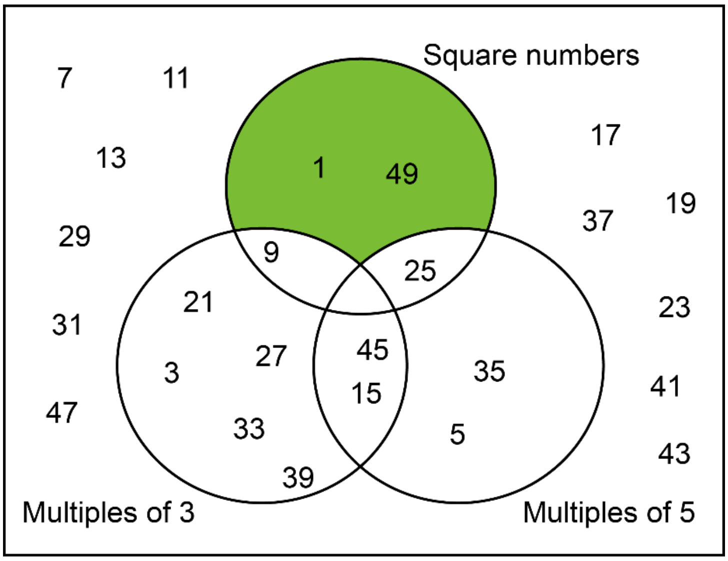
To find the probability of one of these numbers being a square number, a multiple of 3 or a multiple of 5, we need to identify how many of these numbers are in all three sets. This is the region where all three circles overlap. There are no numbers in this region and so the probability is 0.



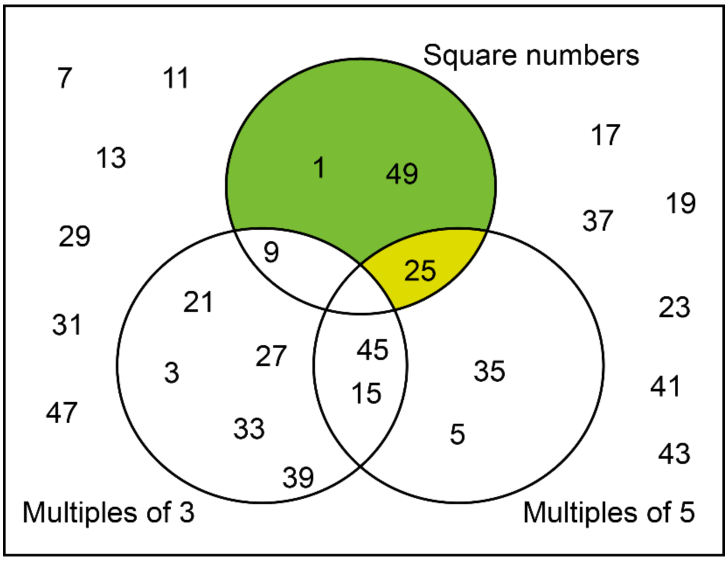
To find the probability that one of these numbers chosen at random is a square number, we need to identify how many numbers are in this set. These are the numbers that are in the top circle of the Venn diagram. There are 4 numbers in this region and so the probability is $\frac{4}{25}$



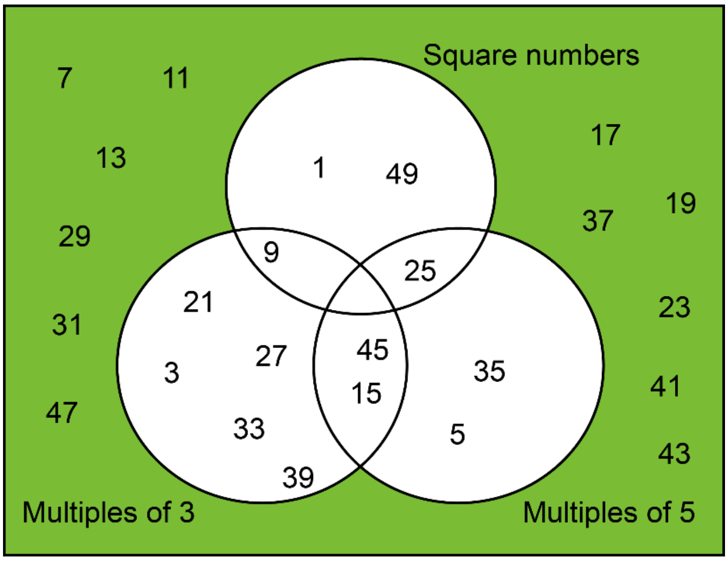
To find the probability that one of these numbers chosen at random would be a square number but not a multiple of 3 or 5, we need to identify how many numbers are in this set. These are the numbers that are in the part of the top circle that does not overlap with either of the other two circles. There are 2 numbers in this region and so the probability is $\frac{2}{25}$



To find the probability that one of these numbers chosen at random is a square number but not a multiple of 3, we need to identify how many numbers are in this set. These are the numbers that are in the part of the top circle that does not overlap with the circle for multiples of 3. There are 3 numbers in this region and so the probability is $\frac{3}{25}$



To find the probability that one of these numbers chosen at random is not a square number, a multiple of 3 or a multiple of 5, we need to identify how many numbers are in this set. These are the numbers that are not contained within any of the three circles. There are 12 numbers in this region and so the probability is $\frac{12}{25}$



e) The first number to be in the centre of the Venn diagram (where all three circles overlap) will be the first odd number that is a square number, a multiple of 3 and a multiple of 5.

We can find this number by working out $(3 \times 5)^2 = 15^2 = 225$

Using probability trees

A fair three-sided spinner can land on red (R) or on yellow (Y) or on blue (B).

Draw a tree diagram to represent the possible results of spinning the spinner twice.

Use the tree diagram to list the possible outcomes of spinning the spinner twice.

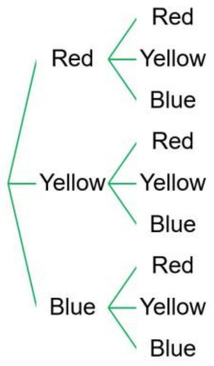
What is the probability of the spinner landing on the same colour twice?

What is the probability of the spinner landing on yellow and blue (in either order)?

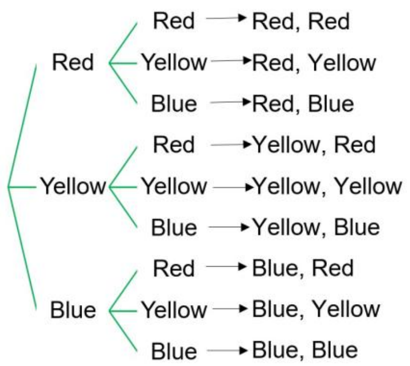
What is the probability that the spinner does not land on yellow on either spin?

What is the probability that the spinner lands on red at least once in the two spins?

We can draw a tree diagram by drawing three branches for the first spin – one for each of red, yellow and blue –and then drawing a second set of branches for the second spin.



We can list the possible outcomes of the pair of spins by reading along the branches of the tree diagram. The possible outcomes are:



There are three of the possible outcomes that have the spinner landing on the same colour twice. There are a total of 9 possible outcomes. P(same colour twice) = $\frac{3}{9} = \frac{1}{3}$

There are two of the possible outcomes that have yellow and blue (in either order). There are a total of 9 possible outcomes.

P(yellow and blue in either order) = $\frac{2}{9}$

There are four outcomes where the spinner does not land on yellow in either spin. There are a total of 9 possible outcomes.

P(does not land on yellow in either spin) = $\frac{4}{9}$

There are five outcomes where the spinner lands on red at least once. There are a total of 9 possible outcomes. P(lands on red at least once) = $\frac{5}{9}$

Using probability trees

A coin is tossed three times in a row and the result of each toss is recorded.

Draw a probability tree to represent all of the possibilities.

List the set of possible outcomes of the three tosses.

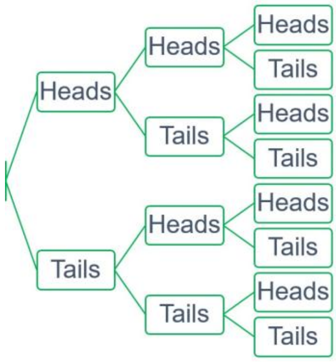
What is the probability of:

no heads?

exactly two heads?

at least one head?

We draw a set of branches for each of the three coin tosses. Each coin toss has heads and tails as a possible outcome.



We can list the possible outcomes of the three coin tosses by reading along the branches to see the possible combinations.

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

One of the possible outcomes of the three coin tosses has no heads. This is TTT. There are a total of 8 possible outcomes. P(no heads) = $\frac{1}{8}$

There are 3 possible outcomes which have exactly two heads. These are HHT, HTH and THH. There are a total of 8 possible outcomes. P(exactly two heads) = $\frac{3}{8}$

To find the probability of at least one head, we can use the same approach as with the two previous parts or we can use the sum of mutually exclusive exhaustive events as being 1.

P(at least one head) + P(no heads) = 1

P(at least one head) = 1 − $\frac{1}{8} = \frac{7}{8}$

M7.6

Construct theoretical possibility spaces for single and combined experiments with equally likely outcomes, and use these to calculate theoretical probabilities.

Possibility spaces (or sample spaces) are used to represent the different possible outcomes for probability experiments with equally likely outcomes. This can then be used to work out the theoretical probability of events occurring.

These can be used for single experiments or combined experiments.

For example, the possible outcomes of flipping a coin are heads or tails. These are equally likely outcomes (provided the coin is fair) and so the probability of obtaining a head on a single flip of a fair coin is $\frac{1}{2}$. The probability of obtaining a tail on a single flip of a fair coin is also $\frac{1}{2}$

Here is an example of a possibility space (sample space) diagram for rolling two unbiased six-sided dice each with sides numbered from 1 to 6.

	1	**2**	**3**	**4**	**5**	**6**
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

This shows that there are 36 possible outcomes of the two dice rolls. We can use this diagram to find the probability of different events. For example, the probability of both dice showing an odd number is $\frac{9}{36} = \frac{1}{4}$

Single experiments

List the possible outcomes of rolling a normal six-sided die.

Assuming that the die is fair, what is the theoretical probability that on a single roll the die lands on:

a three?

an odd number?

a five or more?

a square number?

We need to list all of the possible values that the die could land on. These are 1, 2, 3, 4, 5, and 6.

P(three) $= \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \frac{1}{6}$

P(odd number) $= \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \frac{3}{6} = \frac{1}{2}$

This is because there are three odd numbers that could be rolled: 1, 3, 5.

P(five or more) = $\frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \frac{2}{6} = \frac{1}{3}$

This is because there are two numbers which are 5 or more: 5 or 6.

P(square number) = $\frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \frac{2}{6} = \frac{1}{3}$

This is because there are two numbers which are square numbers: 1 or 4.

Combined experiments

Two fair six-sided dice are rolled, and the scores shown are multiplied together.

Draw a possibility space diagram to show the possible outcomes.

What is the probability of the product of the two scores being

odd?

greater than 10 but less than 20?

As we are interested in the product of the two dice rolls, we can fill the possibility space with the products of the rolls rather than the possible pairs of outcomes.

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

There are 9 odd products out of the 36 possible outcomes: $\frac{9}{36} = \frac{1}{4}$

There are 9 products out of the 36 possible outcomes which are greater than 10 but less than 20: $\frac{9}{36} = \frac{1}{4}$

M7.7

Know when to add or multiply two probabilities, and understand conditional probability.

Calculate and interpret conditional probabilities through representation using expected frequencies with two-way tables, tree diagrams and Venn diagrams.

Understand the use of tree diagrams to represent outcomes of combined events:

a. when the probabilities are independent of the previous outcome
b. when the probabilities are dependent on the previous outcome.

Addition of probabilities

Events are said to be mutually exclusive if no more than one of the events can occur at any time.

When two events, A and B, are mutually exclusive, the probability that A or B will occur is the sum of the probability of each event.

P(A or B) = P(A) + P(B)

When two events, A and B, are not mutually exclusive, there is overlap between the two events – they can occur at the same time. The probability that A or B will occur is the sum of the probability of each event minus the probability of the overlap.

P(A or B) = P(A) + P(B) − P(A and B)

Multiplication of probabilities

Two events, A and B, are independent if the fact that A occurs does not affect the probability of B occurring.

If two events are independent then we can find the probability of A and B occurring by multiplying the probabilities of the two events.

P(A and B) = P(A) × P(B)

We can use tree diagrams to calculate combined probabilities when the events are:

- independent
- not independent.

Conditional probability

If we are interested in the probability of an event B occurring only when it is known that an event A has occurred, then this is called the conditional probability of B given A.

If two events are independent, then the probability of B is equal to the probability of B given A. If the two events are not independent, then these two probabilities will not be the same.

We can calculate conditional probabilities using:

- two-way tables
- Venn diagrams
- tree diagrams.

Addition of probabilities

Alice has a spinner.

When the spinner is spun, the probability that it lands on a square number is $\frac{3}{7}$

When the spinner is spun, the probability that it lands on a prime number is $\frac{1}{3}$

What is the probability that the spinner lands on a prime number or a square number?

Landing on a square number and landing on a prime number are mutually exclusive events as there are no numbers that are both prime numbers and square numbers:

Prime numbers: 2, 3, 5, 7, 11…

Square numbers: 1, 4, 9, 16…

All square numbers will have their square root as a factor and so will not meet the requirement of having exactly two factors to be a prime.

Since landing on a square number and landing on a prime number are mutually exclusive, we can add the two probabilities to find the probability of landing on a prime number or a square number.

P(prime number or square number) = P(prime number) + P(square number):

$= \frac{1}{3} + \frac{3}{7} = \frac{7}{21} + \frac{9}{21} = \frac{16}{21}$

Addition of probabilities

The probability of an event A is 0.32

The probability of an event B is 0.19

The probability of neither event occurring is 0.54 Are the events A and B mutually exclusive?

P(neither event occurring) = 1 − P(A or B occurring)

P(A or B occurring) = 1 − 0.54

= 0.46

P(A or B) = P(A) + P(B) − P(A and B)

0.46 = 0.32 + 0.19 − P(A and B)

P(A and B) = 0.32 + 0.19 − 0.46

= 0.05

The events A and B are not mutually exclusive as P(A and B) ≠ 0

Multiplication of probabilities

The probability of machine A breaking down is 0.4

The probability of machine B breaking down is 0.25

One machine breaking down does not change the probability of the other machine breaking down.

What is the probability that:

both machines break down?

neither machine breaks down?

The two events are independent of each other and so:

P(A breaks down and B breaks down)

= P(A breaks down) × P(B breaks down)

$= 0.4 \times 0.25$

$= 0.1$

The probability that neither machine breaks down can be worked out by multiplying the probability of each machine not breaking down:

P(A does not break down and B does not break down) = P(A does not break down) × P(B does not break down)

= (1 −P(A breaks down)) × (1 −P(B breaks down))

$= (1 - 0.4) \times (1 - 0.25)$

$= 0.6 \times 0.75$

$= 0.45$

Using a tree diagram for independent events

A school has a football team and a cricket team.

The outcomes of the football and cricket matches are win, draw and lose.

The probability that the football team will win its next match is 0.6

The probability that the football team will draw its next match is 0.15

The probability that the cricket team will win its next match is 0.4 The probability that the cricket team will draw its next match is 0.05

Find the probability that:

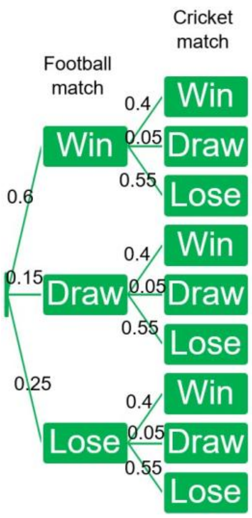
both teams win their next match.

both teams lose their next match.

at least one team wins its next match.

neither team draws its next match.

We can draw a tree diagram to represent the information in the question:



We work out the probability of each team losing by using the fact that the sum of the probabilities of exhaustive mutually exclusive events is 1:

P(football team loses its next match) = 1 − P(football team wins) − P(football team draws) = 1 − 0.6 − 0.15

= 0.25

P(cricket team loses its next match) = 1 − P(cricket team wins) − P(cricket team draws)

= 1 − 0.4 − 0.05 = 0.55

When working out the probability of a particular branch in the probability tree, we multiply along the branches.

When:

Both teams win their next match P(both teams win) = $0.6 \times 0.4 = 0.24$

Both teams lose their next match

P(both teams lose their next match) = $0.25 \times 0.55 = 0.1375$

At least one team wins its next match

P(at least one team wins its next match) = $0.6 \times 0.4 + 0.6 \times 0.05 + 0.6 \times 0.55 + 0.15 \times 0.4 + 0.25 \times 0.4 = 0.76$

(There is no need to calculate $0.6 \times 0.4 + 0.6 \times 0.05 + 0.6 \times 0.55$ as this equals 0.6. If the football team wins, that satisfies the condition.)

Neither team draws its next match

P(neither team draws) = $0.6 \times 0.4 + 0.6 \times 0.55 + 0.25 \times 0.4 + 0.25 \times 0.55$

$= 0.8075$

Using a tree diagram for independent events

A bag contains red counters and blue counters only.

There are 7 red counters and 3 blue counters in the bag.

A counter is picked at random from the bag, its colour noted, and returned to the bag.

A second counter is then picked at random from the bag.

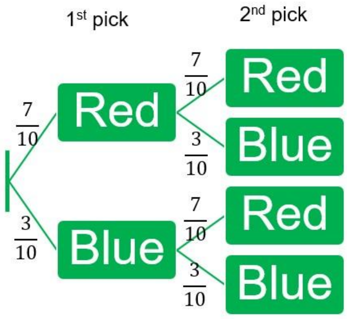
What is the probability that:

both counters are red?

at least one counter is blue?

one counter of each colour is picked?

We can draw a tree diagram to help with this question.



P(both counters are red) $= \frac{7}{10} \times \frac{7}{10} = \frac{49}{100}$

P(at least one counter is blue) $= \frac{7}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{7}{10} + \frac{3}{10} \times \frac{3}{10} = \frac{51}{100}$

or

P(at least one counter is blue) = 1 −P(both red)

$= 1 - \frac{49}{100}$

$= \frac{51}{100}$

P(one counter of each colour is picked) $= \frac{7}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{7}{10} = \frac{41}{100} = \frac{21}{50}$

Using a tree diagram for dependent events

A bag contains red counters and blue counters only.

There are 7 red counters and 3 blue counters in the bag.

A counter is picked at random from the bag and is not replaced.

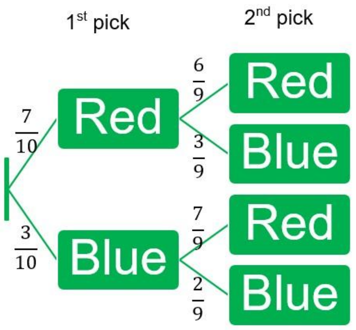
A second counter is then picked at random from the bag.

What is the probability that:

both counters are red?

at least one counter is blue?

one counter of each colour is picked?



P(both counters are red) $= \frac{7}{10} \times \frac{6}{9} = \frac{42}{90} = \frac{14}{30} = \frac{7}{15}$

P(at least one counter is blue) $= \frac{7}{10} \times \frac{3}{9} + \frac{3}{10} \times \frac{7}{9} + \frac{3}{10} \times \frac{2}{9} = \frac{48}{90} = \frac{24}{45} = \frac{8}{15}$

or

P(at least one counter is blue) = 1 − P(both red)

$= 1 - \frac{7}{15}$

$= \frac{8}{15}$

P(one counter of each colour is picked) $= \frac{7}{10} \times \frac{3}{9} + \frac{3}{10} \times \frac{7}{9} = \frac{42}{90} = \frac{21}{45} = \frac{7}{15}$

Conditional probabilities – two-way table

The two-way table shows the number of workers in a factory who are employed working on the production line, in the warehouse and in the office.

It also shows the number of workers who drive to work and the number of workers who do not drive to work.

	Production line	Warehouse	Office	Total
Drive to work	35	37	21	93
Do not drive to work	18	20	19	57
Total	53	57	40	150

A worker is chosen at random to take part in a survey.

Given that the worker is from the warehouse, what is the probability that they drive to work?

Given that the worker drives to work, what is the probability that they work in the office?

P(drive to work given they work in the warehouse) =

$$\frac{\text{number of people who drive to work who work in the warehouse}}{\text{number of people who work in the warehouse}}$$

$= \frac{37}{57}$

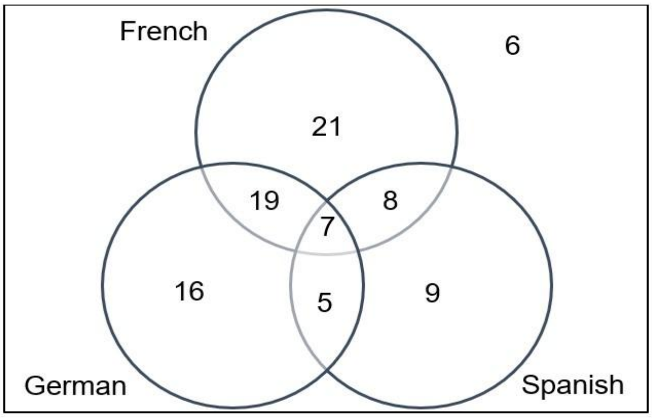
P(work in the office given they drive to work) =

$$\frac{\text{number of people who work in the office who drive to work}}{\text{number of people who drive to work}}$$

$= \frac{21}{93} = \frac{7}{31}$

Probabilities including conditional probabilities – Venn diagram

The Venn diagram shows the number of students in Year 11 at a secondary school who study French, German or Spanish.



A student from Year 11 is picked at random. What is the probability that:

the student studies one language only?

the student does not study German?

given that the student studies German, they also study French?

given that the student studies only one language, that language is French?

P(studies one language only) =

$$\frac{\text{number of students who study one language only}}{\text{total number of students}}$$

$$= \frac{21+9+16}{21+19+7+8+16+5+9+6} = \frac{46}{91}$$

P(does not study German) =

$$\frac{\text{number of students who do not study German}}{\text{total number of students}}$$

$$= \frac{21+8+9+6}{91} = \frac{44}{91}$$

P(studies French given they study German) =

$$\frac{\text{number of students who study French and German}}{\text{number of students who study German}}$$

$$= \frac{19+7}{16+5+19+7} = \frac{26}{47}$$

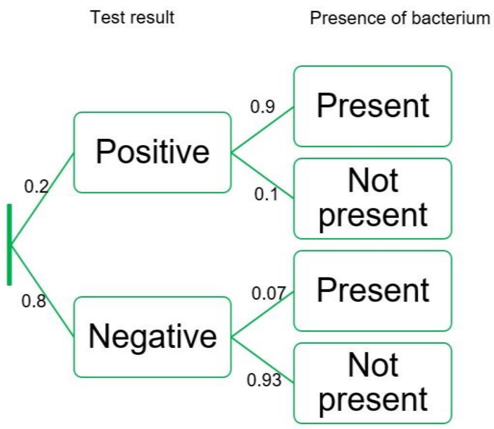
P(studies French given they study only one language) =

$$\frac{\text{number of students who study one language and that language is French}}{\text{number of students who study only one language}}$$

$$= \frac{21}{21+16+9} = \frac{21}{46}$$

Conditional probabilities – tree diagram

The tree diagram below shows the probability of a test for a particular bacterium being positive, and the probability that the bacterium is actually present.



Given that the bacterium is present, what is the probability of a positive test result?

P(positive test result given bacterium present) =

$$\frac{P(\text{positive test result and bacterium present})}{P(\text{bacterium present})}$$

$$\frac{0.2\times0.9}{0.2\times0.9+0.8\times0.07}$$

= 0.763 (3 decimal places)